Asset pricing in the frequency domain: theory and empirics

Ian Dew-Becker and Stefano Giglio

Duke Fuqua and Chicago Booth

11/27/13

Introduction

- What risks are people willing to pay to avoid? What determines risk premia?
  - Early models: what's happening right now (CAPM, CCAPM)
  - Later models: news about the future (EZ, ICAPM)
- Risk prices depend on **dynamics** of shocks
- In this paper we derive a new **frequency-domain representation** of asset prices
  - Frequency domain is the natural place to study implications of dynamics

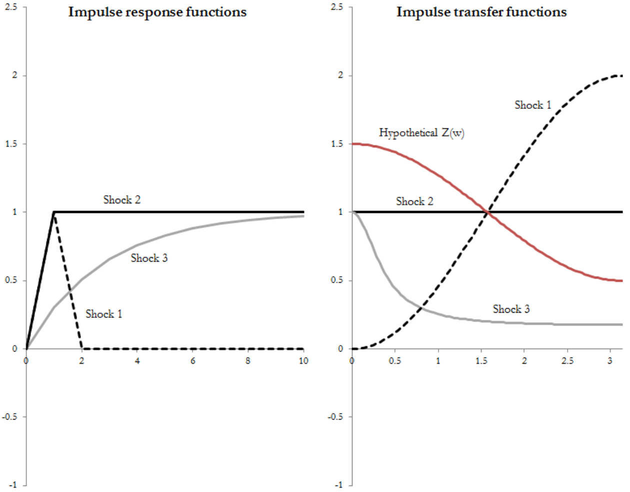
Introduction

- Spectral decomposition of risk prices in all affine models
  - Consumption as an example today
- Separates preferences from dynamics
  - Difference from Hansen et al.
- Allows non-parametric estimation
  - *Not possible* in time domain
  - Focus on economically important frequencies

What we do

- Suppose agents care about current and future consumption growth
- When a fundamental shock ε hits it moves consumption now and in the future
- The **impulse-response function** to ε tells us *how* it moves consumption
- Alternatively: ε *induces many fluctuations in consumption*
  - These "sum up" to the total response of consumption
- $G_\varepsilon(\omega)$ captures ω-frequency response: **impulse transfer function**
- It's the Fourier transform of the IRF

Which shock has the largest risk premium?



Why this is useful

1. See how **dynamics** are priced in different models
   - Make statements about "long-run risk", low- vs. high-frequency **rigorous**
2. Non-parametric estimation
   - Test **economic intuition** behind models instead of strict parametrizations

What we find: Theory

- Preferences have strong implications for $Z(\omega)$
  - Power utility: flat $Z(\omega)$
  - Internal habits: high weight on high frequencies
  - Epstein–Zin: power at *lowest* frequencies (230 years on avg.)
  - Simple test: slope of Z
- Standard models surprisingly restrictive
  - All functions are monotone
  - Cannot isolate business-cycle frequencies

What we find: Empirics

- Structural Epstein–Zin fails
  - No significant coefficients
  - "Long-run" is too long
  - Implies consumption does not price equities
- Frequency domain generalization works
  - Define "long-run" as cycles longer than 8 years
  - Long-run shocks are significantly priced

- Results apply to affine models generally
  - Not just consumption-based models
- Related papers:
  - Term structure of interest rates
  - VIX futures curve

Outline

1. Theory
   1. Basic frequency decomposition
   2. Application to utility functions
2. Empirics; estimate the weighting function

Asset pricing

- With no arbitrage, there exists a stochastic discount factor (SDF),

$$1 = E_t \left[R_{t+1} M_{t+1} \right]$$

  for all returns R_{t+1}
- Excess returns depend on covariances with the SDF,

$$\begin{aligned} E_t \left(R_{t+1} - R_{f,t+1} \right) &= -cov \left(R_{t+1}, \frac{M_{t+1}}{E_t M_{t+1}} \right) \\ &\approx -cov \left(r_{t+1}, m_{t+1} \right) \end{aligned}$$

$r = \log R$, $m = \log M$

- What moves M_{t+1}, and by how much?

Affine models

- **Assumption 1:** Log SDF (m_{t+1}) depends on the dynamics of a state variable x_t

$$m_{t+1} - E_t m_{t+1} = \sum_{k=0}^{\infty} z_k \Delta E_{t+1} x_{t+k+1}$$
$$\text{where } \Delta E_{t+1} \equiv E_{t+1} - E_t$$

  E_t is expectation operator; $\{z_k\}$ a set of known weights

  - Common models: x_t is consumption growth or equity returns
  - Epstein–Zin: innovation to SDF depends on future consumption growth (long-run risk)

- **Assumption 2:** x_t driven by a VMA process

$$x_t = B_1 \Gamma(L) \varepsilon_t$$

for a vector of innovations ε_t, selection vector $B_1 = [1, 0, 0, ...]$, $\Gamma(L)$ lag polynomial,

$$\Gamma(L) = \sum_{k=0}^{\infty} \Gamma_k L^j$$

- We *do not* assume anything about higher moments:
  - Normality
  - Homoskedasticity

Standard risk prices

- Impulse response function:

$$g_{j,k} = \frac{dE_t x_{t+k}}{d\varepsilon_{j,t}}$$

- Innovations to the SDF are

$$m_{t+1} - E_t m_{t+1} = -\sum_j \left(\sum_{k=0}^{\infty} z_k g_{j,k} \right) \varepsilon_{j,t+1}$$

  where $\{g_{j,k}\}$ is the IRF of x to ε_j
- Risk price for ε_j is $\sum_{k=0}^{\infty} z_k g_{j,k}$

- **Result 1:** Under assumptions 1 and 2,

$$m_{t+1} - E_t m_{t+1} = -\sum_j \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} Z(\omega)\, G_j(\omega)\, d\omega \right) \varepsilon_{j,t+1}$$

- Risk price for ε_j is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} Z(\omega)\, G_j(\omega)\, d\omega = \sum_{k=0}^{\infty} z_k g_{j,k}$$

- $G_j(\omega)$ is the impulse transfer function, frequency analog of IRF

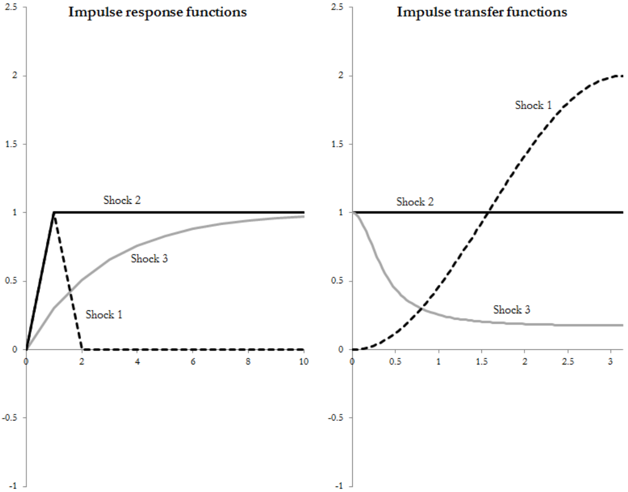
$$G_j(\omega) \equiv \sum_{k=0}^{\infty} \cos(\omega k)\, g_{j,k}$$

- $Z(\omega)$ is the unique price of risk at frequency ω

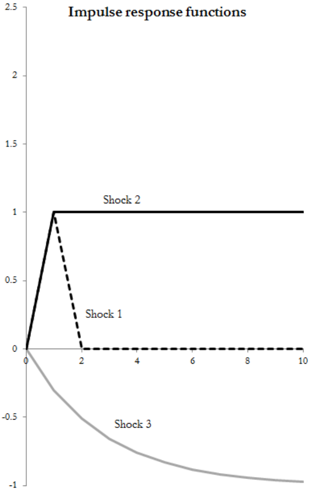
$$Z(\omega) \equiv z_0 + 2 \sum_{k=1}^{\infty} z_k \cos(\omega k)$$

- Z and G_j separate preferences and dynamics

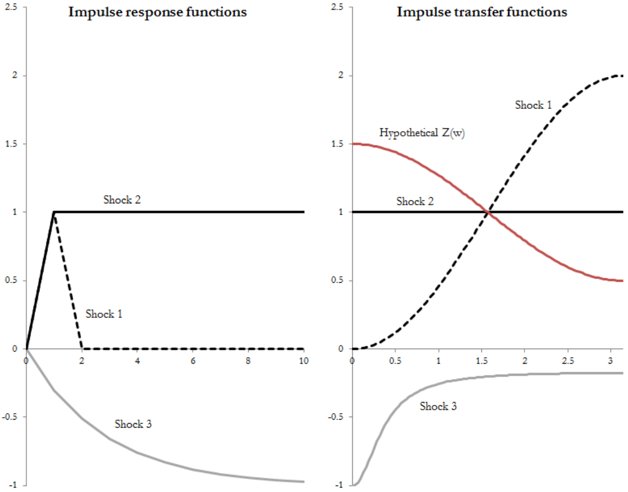
Examples of IRFs and impulse transfer functions



Examples of IRFs and impulse transfer functions



Examples of IRFs and impulse transfer functions



Outline

1. Theory
   1. Basic frequency decomposition
   2. **Application to utility functions**
2. Empirics; estimate the weighting function

Example 1: power utility

- Log pricing kernel is exactly

$$m_{t+1} - E_t m_{t+1} = -\alpha(\Delta c_{t+1} - E_t \Delta c_{t+1})$$

- Weights are $z_0 = \alpha$, $z_k = 0$ for $k > 0$

$$Z^{power}(\omega) = \alpha$$

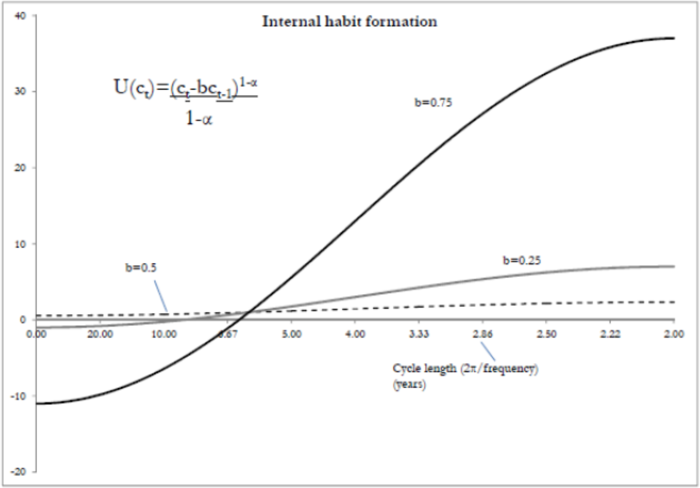
- All shocks have equal weight no matter how long they last

Example 2: Internal habits

- Period utility:

$$U(C_t) = \frac{(C_t - bC_{t-1})^{1-\alpha}}{1-\alpha}$$

- b determines size of habit (and risk aversion)
- We log-linearize the SDF to get weighting function


Internal habit formation
$U(c_t)=\frac{(c_t-bc_{t-1})^{1-\alpha}}{1-\alpha}$
b=0.75
b=0.5
b=0.25
40
30
20
10
0
-10
-20
0.00
20.00
10.00
6.67
5.00
4.00
3.33
2.86
2.50
2.22
2.00
Cycle length (2π/frequency)
(years)

Example 3: External habits

- Period utility:

$$U(C_t) = \frac{\left(C_t - b\bar{C}_{t-1}\right)^{1-\alpha}}{1-\alpha}$$

- $\bar{C}$ is aggregate consumption

$$M_{t+1} = \left(\frac{C_{t+1} - b\bar{C}_t}{C_t - b\bar{C}_{t-1}}\right)^{-\alpha}$$

- Future dynamics do not matter – marginal utility depends only on today's consumption
- Weighting function is flat – like power utility

Example 4: Epstein–Zin preferences

- Assume homoskedastic, log-normal consumption growth
- SDF can be written as

$$\Delta E_{t+1} m_{t+1} = -\left(\rho \Delta E_{t+1} \Delta c_{t+1} + (\alpha - \rho) \Delta E_{t+1} \sum_{j=0}^{\infty} \theta^j \Delta c_{t+1+j} \right)$$

[α risk aversion; ρ inverse EIS; θ linearization parameter near 1]

- $(\alpha - \rho)$ is long-run risk term

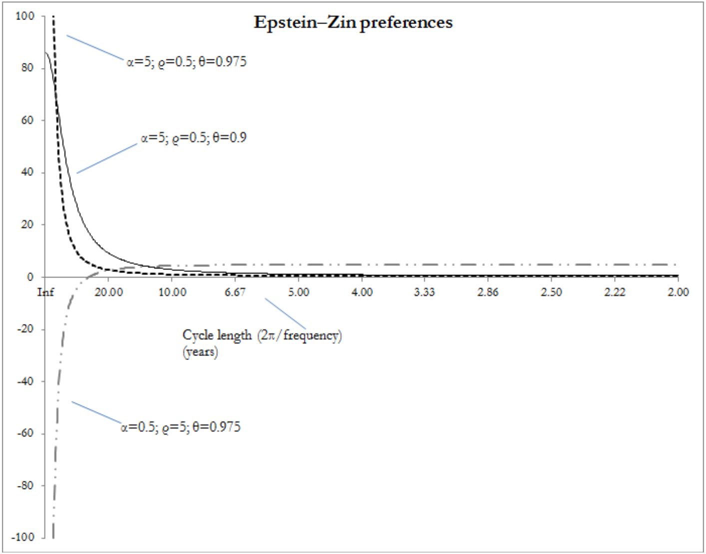
$$Z(\omega) = \alpha + 2(\alpha - \rho) \sum_{j=1}^{\infty} \theta^j \cos(\omega j)$$

- Total mass of Z is α

- For $\theta \to 1$, Z approaches

$$Z(\omega) = (\alpha - \rho)\,\delta(\omega) + \rho$$

- $\delta(\omega)$ is a point mass at zero (periodic extention of Dirac δ)
- Standard calibrations say primarily frequency zero matters
  - $(\alpha - \rho)/\alpha \approx 1$



Epstein–Zin preferences
100
80
60
40
20
0
-20
-40
-60
-80
-100
α=5; ϱ=0.5; θ=0.975
α=5; ϱ=0.5; θ=0.9
α=0.5; ϱ=5; θ=0.975
Inf
20.00
10.00
6.67
5.00
4.00
3.33
2.86
2.50
2.22
2.00
Cycle length (2π/frequency)
(years)

- Models have strong, surprising implications for $Z(\omega)$
  - What do we mean by "long run"?
  - EZ preferences have more than half the weight on cycles longer than **230 years**!
  - Interpretation: take a permanent consumption shock. Half of its price comes from cycles >230 years. $\frac{3}{4}$ of its price from cycles >75 years.
- Clear, sharp differences between E–Z, power, habit formation utility

Implications for calibration

- Models are usually calibrated to match unconditional moments
- With a non-trivial weighting function, consumption dynamics matter
- For internal habits, autocorrelation matters
- For Epstein–Zin, need to calibrate long-run standard deviation
    - Std. dev. of innovations to Beveridge–Nelson trend
    - LRR models sometimes as high as 4% per quarter
    - Empirically, no more than 2% per quarter

A limitation

- All the weighting functions are monotone
- Maybe consumers dislike mainly business-cycle frequency shocks?
  - Huge policy literature suggests BC is relevant
- Standard models do not allow that

Multiple priced variables

- What about higher moments?
  - E.g. priced disaster risk or volatility shocks
- Price of risk for a shock ε_j is

$$\sum_m \frac{1}{2\pi} \int_{-\pi}^{\pi} Z_m(\omega) G_{m,j}(\omega) d\omega$$

  m indexes priced state variables
- Each priced variable gets a weighting function $Z_m(\omega)$
  - For Epstein–Zin, $Z_m(\omega)$ is almost always isolated near zero (only long-run volatility shocks should be priced)

Outline

1. Theory
    1. Basic frequency decomposition
    2. Application to utility functions
2. **Empirics; estimate the weighting function**

Estimation

- We don't necessarily need FD for estimation
  - Anything done in the FD, in principle, works in time domain
- Why is the frequency decomposition useful?
  - Generalize models
  - Parameterize estimation in terms of frequencies directly (*not possible* in time domain)

- Two-step estimation:
    - Estimate reduced-form risk prices
    - Rotate into frequency domain (using IRF)
- State variables follow a VAR

$$X_t = \Phi X_{t-1} + \varepsilon_t$$

- Consumption growth is first element of X
- Remainder of X should forecast consumption growth
- Rotate risk prices on ε_t

$$X_t = \Phi X_{t-1} + \varepsilon_t$$

- Estimate reduced-form prices for ε_t $(\bar{p})$
  - GMM on cross-section of equity returns $(1 = E[MR])$
- Each ε_t has an ITF $G_k(\omega; \Phi)$; risk prices are

$$\bar{p}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} Z(\omega) G_k(\omega; \Phi) \, d\omega$$

- With K shocks, can estimate Z up to K degrees of freedom
  - Need basis functions for $Z(\omega)$

Parametrizing Z: the utility basis

- Models we have explored before:

$$Z^U(\omega) = q_1 \sum_{j=1}^{\infty} \theta^j \cos(\omega j) + q_2 + q_3 \cos(\omega)$$

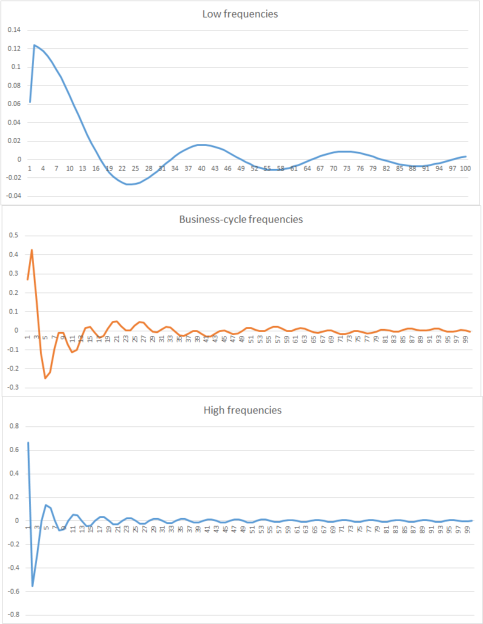
- If $q_3 = 0$, we have EZ
- If $q_1 = 0$, we have internal habit
- If $q_1 = q_3 = 0$, we have power utility
- Note: we have an extra parameter θ.
  - Poorly identified, so use standard calibration (0.975)

Parametrizing Z: the bandpass basis

- Groups together frequencies
- Write $Z(\omega)$ directly as a step function.

$$Z^{BP}(\omega) = q_1 Z^{(0,2\pi/32)}(\omega) + q_2 Z^{(2\pi/32,2\pi/6)}(\omega) + q_3 Z^{(2\pi/6,\pi)}(\omega)$$

- Three components
  - lower-than-BC: $>$ 32 quarters
  - BC: 6-32 quarters
  - higher-than-BC: $<$6 quarters


Low frequencies
Business-cycle frequencies
High frequencies

Rotating the risk prices

- SDF is

$$\Delta E_{t+1} m_{t+1} = \left(\sum_{j=0} z_k B_1 \Phi^j \right) \varepsilon_t$$

$$= \underbrace{\overbrace{\bar{q}'}^{\text{Frequency-domain risk prices}} \times \overbrace{\left(\sum_{j=0} \begin{bmatrix} \frac{1}{\pi} \int_{-\pi}^{\pi} Z_1(\omega) \cos(\omega k)\, d\omega \\ \frac{1}{\pi} \int_{-\pi}^{\pi} Z_2(\omega) \cos(\omega k)\, d\omega \\ \frac{1}{\pi} \int_{-\pi}^{\pi} Z_3(\omega) \cos(\omega k)\, d\omega \end{bmatrix} B_1 \Phi^j \right)}^{\text{Rotation into Frequency domain}}}_{\text{Reduced-form risk prices}} \varepsilon_t$$

1. Estimate dynamic effects of shocks on consumption growth (Φ, IRF)
2. Estimate reduced-form risk prices on ε_t ($\bar{p}$)
3. Choose basis functions
4. Rotate $\bar{p}$ into frequency domain

Empirics: Data

- 25 size- and B/M-sorted portfolios; 49 industry portfolios, post-war data
- Priced variables: Consumption, GDP, investment components
  - Cochrane (1996) argues investment priced
- Predictive variables (X):
  - Two principal components of 13 standard predictors:
  - P/E; P/D; term spread; default spread; unemployment; value spread; short-term interest rate; equity issuance; I/K; *cay*

Empirics: Estimation

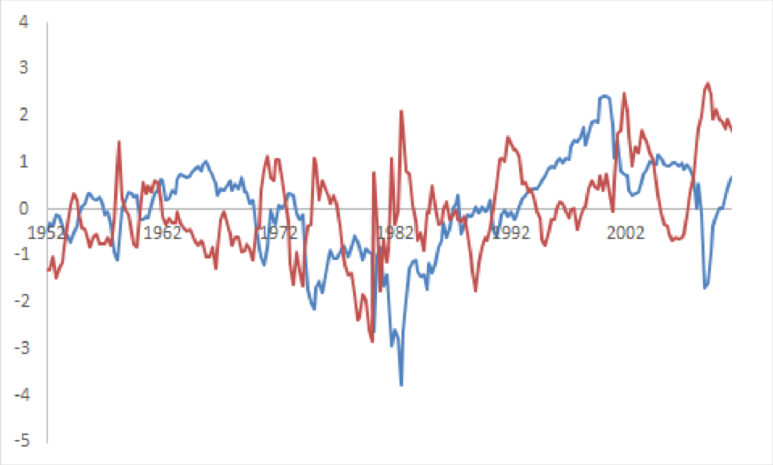
1. Estimate VAR, $X_t = \Phi X_{t-1} + \varepsilon_t$
2. Estimate reduced-form prices of ε_t ($\bar{p}$)
   - Use moment condition

$$0 = E\left[\left(R_{i,t+1} - R_{f,t+1}\right)\exp\left(-\bar{p}\varepsilon_{t+1}\right)\right]$$

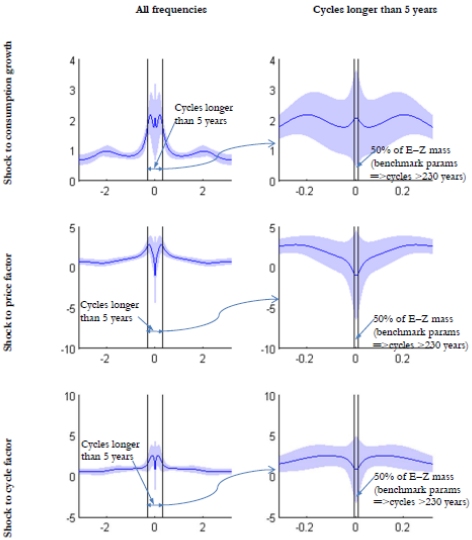
3. Rotate $\bar{p}$ to get parameters in spectral weighting function

Parameters are estimated in two stages
We construct standard errors from combined GMM moment

Predictive variables



Estimated impulse transfer functions



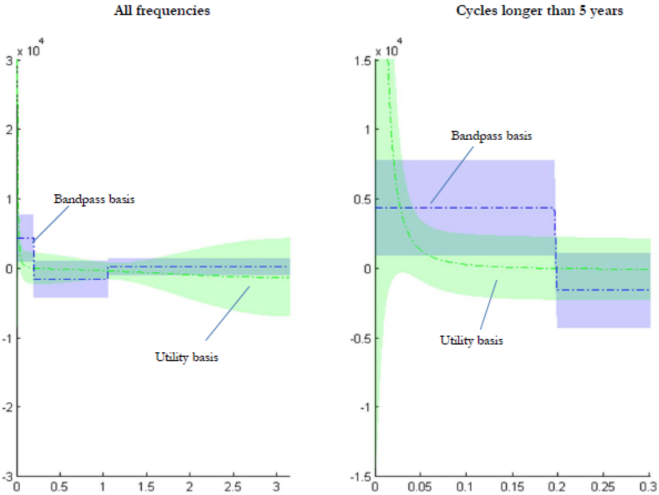
- Immediately clear it will be difficult/impossible to estimate Epstein–Zin
- Need more power at low frequencies
- Could impose theoretical restrictions
    - Cointegration
    - Similar to Blanchard and Quah (1989)
    - Still must estimate long-run behavior of *something*

Main estimates

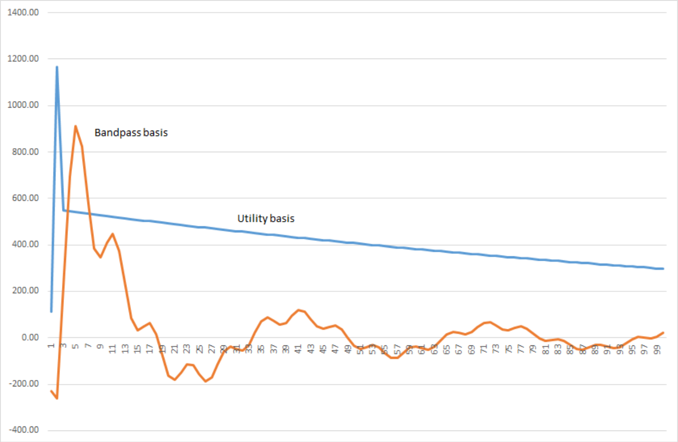
Portfolios:		FF25				FF25+IND			
Basis:		Bandpass	t-stat	Utility (0.975)	t-stat	Bandpass	t-stat	E–Z (0.975)	t-stat
Consumption growth	q1	269	2.47 **	555.47	1.66 *	112	1.95 *	197.52	1.35
	q2	-431	-1.17	-442.65	-0.44	-116	-0.87	-279.30	-0.65
	q3	138	0.33	616.12	0.32	-134	-0.70	504.32	0.63

- Weak results with utility basis – no frequencies seem to matter
- Highly significant results for bandpass basis
  - Long-run shocks significantly priced
- Coefficients map to risk aversion
- Results robust to using one-step GMM (though weaker)

Estimated weighting functions



Time-domain weights



Equity Pricing: results

- Notes on the estimation
  - We have used the efficient matrix for the asset pricing conditions (GMM)
  - Results robust to including Industry portfolios, and using identity matrix for GMM
- Focusing on frequency domain allowed us to:
  - Estimate a more economically intuitive version of “long-run”
  - Focus on important frequencies for which we have power ($<$230 years!)

Alternative state variables

Basis:		Bandpass	t-stat	Utility (0.975)	t-stat
Consumption growth	q1	269	2.47 **	555.47	1.66 *
	q2	-431	-1.17	-442.65	-0.44
	q3	138	0.33	616.12	0.32
GDP	q1	124	1.85 *	231.42	0.69
	q2	-106	-1.29	119.67	0.87
	q3	127	1.33	-217.42	-1.04
Durables	q1	49	2.66 ***	75.62	1.69 *
	q2	-38	-1.26	44.87	2.29 **
	q3	33	1.70 *	-86.66	-0.63
Investment	q1	12	2.03 **	29.25	1.03
	q2	-7	-1.18	-0.22	-0.03
	q3	-7	-1.12	5.17	0.36
Fixed Investment	q1	27	2.16 **	39.33	0.81
	q2	-25	-1.11	67.18	2.59 ***
	q3	61	3.10 ***	-90.96	-1.66 *
Residential Investment	q1	16	3.52 ***	27.00	2.44 **
	q2	-3	-0.45	-4.74	-0.12
	q3	4	0.24	55.19	0.85

- Utility: weak, erratic results
- Bandpass: low frequencies priced for all variables
  - Consistent with PIH
  - Doesn't distinguish utility models

Estimating structural models is treacherous

- Epstein–Zin says investors only care about frequency zero
- This is not testable; makes the model look useless
- Allowing more frequencies to be priced helps

Returns-based models

- Pricing kernel also stated in terms of returns
- CAPM:

$$\Delta E_{t+1} m_{t+1} \approx -\frac{E\left[r^m\right]}{Var\left[r^m\right]} \Delta E_{t+1} r^m_{t+1}$$

- Epstein–Zin, power utility (Campbell, 1993):

$$\Delta E_{t+1} m_{t+1} = -\alpha \Delta E_{t+1} r_{w,t+1} + (1-\alpha) \Delta E_{t+1} \sum_{j=1}^{\infty} r_{w,t+j+1}$$

  - Two factor model: current returns, discount-rate news
- So use returns as priced variable instead of macro aggregates

Estimates with returns as priced variable

		coeff	t-stat
Utility basis	q1 Long-run	8.36	2.19 **
	q2 Constant	9.99	3.34 ***
Bandpass basis	q1 Long-run	13.10	2.36 **
	q2 Constant	-2.36	-0.65

Related literature

- **Hansen and Sheinkman (2009), Hansen and Borovicka (2012):** How m_{t+1} **evolves** following a shock (IRF of m)
  - We study how the 1-period innovation in m depends on the evolution of consumption after a shock
- **Hansen, Heaton and Li (2008)**, **Lettau and Wachter (2007)**: Price zero-coupon dividend claims
  - Combines cash-flow and SDF dynamics, (link $E_t m_{t+1}$ to Δc_{t+1})
- **Alvarez and Jermann (2005):** Permanent vs. transitory component of the pricing kernel

Related literature

We focus on what drives the volatility of SDF today

- **Disadvantages**: we don't price zero-coupon claims
- **Advantages:**
  - It's all you need to price returns: $0 = E_t[R^e_{t+1}(M_{t+1} - E_t M_{t+1})]$
  - Study separately preferences and dynamics
  - To study evolution of the whole m you need to specify how the risk-free rate evolves, which depends on the dynamics of consumption.

Conclusion

- We derive a frequency-domain representation of affine asset pricing models
  - Assumptions required (existence of MA representation, SDF depends on dynamics) are standard and minimal
- 1) Obtain sharp implications in many models
- 2) Allows us to easily generalize models and focus on economically relevant frequencies