

# The price of variance risk

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## Abstract

In the period 1996–2014, it was costless on average to hedge news about future variance at horizons ranging from 1 quarter to 14 years. It is only purely transitory and unexpected realized variance that was priced. These results present a challenge to many structural models of the variance risk premium, such as the intertemporal CAPM, recent models with Epstein–Zin preferences and long-run risks, and models where institutional investors have value-at-risk constraints. The results are also difficult to reconcile with macro models in which volatility affects investment decisions.

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# 1 Introduction

The recent explosion of research on the effects of volatility in macroeconomics and finance shows that economists care about uncertainty shocks. It appears that investors, on the other hand, do not. In the period since 1996, it has been costless on average to hedge news about future volatility in aggregate stock returns; in other words investors have not been required to pay for insurance against volatility news. Many economic theories – both in macroeconomics and in finance – have the opposite prediction. The recent consumption-based asset pricing literature is heavily influenced by Epstein–Zin (1991) preferences, which in standard calibrations with preference for early resolution of uncertainty imply that investors have a strong desire to hedge news about future uncertainty, and hence should be willing to pay large premia for insurance against volatility shocks. Furthermore, in recent macroeconomic models shocks to uncertainty about the future can induce large fluctuations in the economy.<sup>1</sup> But if increases in economic uncertainty can drive the economy into a recession, we would expect that investors would want to hedge those shocks. The fact that shocks to expected volatility have not earned a risk premium thus presents a challenge to a wide range of recent research.

As a concrete example, consider the legislative battles over the borrowing limit of the United States in the summers of 2010 and 2011. Those periods were associated with increases in both financial measures of uncertainty, e.g. the VIX, and also the measure of policy uncertainty from Baker, Bloom, and Davis (2014). Between July and October, 2011, the 1-month variance swap rate – a measure of investor expectations for S&P 500 volatility over the next month – rose every month, from 16.26 to 42.32 percent (annualized, computed at the beginning of the month). However, those shocks also had small effects on *realized* volatility in financial markets: for example, realized volatility actually decreased between August and September. The debt ceiling debate caused uncertainty about the future to be high during the whole period, but did not correspond to high contemporaneous volatility during the same period. It is precisely this imperfect correlation between realized volatility and expectations of future volatility that allows us to disentangle the pricing of their shocks. In this paper, we directly measure how much people pay to hedge shocks to expectations of future volatility. We find that news shocks have been unpriced: any investor could have bought insurance against volatility shocks for free, and therefore any investor could have freely hedged the increases in uncertainty during the debt ceiling debate.

We measure the price of variance risk using novel data on a wide range of volatility-

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<sup>1</sup>See, e.g., Bloom (2009), Bloom et al. (2014), Christiano, Motto, and Rostagno (2014), Fernandez-Villaverde et al. (2011), and Gourio (2012, 2013)

linked assets both in the US and around the world, focusing primarily on variance swaps with maturities between 1 month and 10 years. The data covers the period 1996–2014. Variance swaps are assets that pay to their owner the sum of daily squared stock market returns from their inception to maturity. They thus give direct exposure to future stock market volatility and are the most natural and direct hedge for the risks associated with increases in aggregate economic uncertainty.

The analysis of the pricing of variance swaps yields two simple but important results. First, news about future volatility is unpriced – exposure to volatility news has not earned a risk premium in our sample. Second, exposure to *realized variance* is strongly priced, with an annualized Sharpe ratio of -1.7 – five times larger than the Sharpe ratio on equities. We find that it is the downside component of realized volatility that investors are specifically trying to hedge, consistent with the results of Bollerslev and Todorov (2011) showing that realized variance is priced due to its correlation with large negative jumps, and also Segal, Shaliastovich, and Yaron (2015), who examine a model with good and bad volatility. We conclude that over our sample, investors paid a large amount of money for protection from extreme negative shocks to the economy (which mechanically generate spikes in realized volatility), but they did not pay to hedge news that uncertainty or the probability of a disaster has changed.

The results present a challenge to a wide range of models. From a finance perspective, Merton’s (1973) intertemporal capital asset pricing model says that assets that have high returns in periods with good news about future investment opportunities are viewed as hedges and thus earn low average returns. Since expected future volatility is a natural state variable for the investment opportunity set, the covariance of an asset’s returns with shocks to future volatility should affect its expected return, but it does not.<sup>2</sup>

Consumption-based models with Epstein–Zin (1991) preferences have similar predictions. Under Epstein–Zin preferences, marginal utility depends on lifetime utility, so that assets that covary positively with innovations to lifetime utility earn high average returns.<sup>3</sup> If high expected volatility is bad for lifetime utility (either because volatility affects the path of consumption or because volatility reduces utility simply due to risk aversion), then volatility

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<sup>2</sup>Recently, Campbell et al. (2014) and Bansal et al. (2013) estimate an ICAPM model with stochastic volatility and find that shocks to expected volatility (and especially long-run volatility) are priced in the cross-section of returns of equities and other asset classes. Although the focus on their paper is not the variance swap market, Campbell et al. (2014) test their specification of the ICAPM model also on straddle returns and synthetic volatility claims, and find that the model manages to explain only part of the returns on these securities. This suggests that the model is missing some high-frequency features of the volatility market.

<sup>3</sup>This is true in the most common calibrations with a preference for early resolution of uncertainty. When investors prefer a late resolution of uncertainty the risk prices are reversed.

news should be priced.<sup>4</sup>

As a specific parameterized example with Epstein–Zin preferences, we study variance swap prices in Drechsler and Yaron’s (2011) calibrated long-run risk model. While that model represents a major innovation in being able to both generate a large variance risk premium (the average gap between the 1-month variance swap rate and realized variance) and match results about the predictability of market returns, we find that its implications for the term structure of variance swap prices and returns are distinctly at odds with the data: it predicts that shocks to future expected volatility should be strongly priced, counter to what we observe empirically.

We obtain similar results in Wachter’s (2013) model of time-varying disaster risk with Epstein–Zin preferences. The combination of fluctuations in the probability of disaster and Epstein–Zin preferences results in a counterfactually high price for insurance against shocks to expected future volatility relative to current volatility. In both Wachter (2013) and Drechsler and Yaron (2011), Sharpe ratios earned by claims on variance more than three months in the future 3 months in the future are similar to those earned by claims to realized variance over the next month, whereas in the data the sample Sharpe ratios are all actually *positive* for claims to variance more than two months in the future. So both models fail to match our key stylized fact that only very short-term variance claims earn large negative Sharpe ratios.<sup>5</sup>

More positively, we show that Gabaix’s (2012) model of rare disasters, which builds on the work of Rietz (1988), Barro (2006), and many others, can match the stylized fact that Sharpe ratios on variance claims fall to zero rapidly with maturity. Intuitively, when investors have power utility, they invest myopically in that they do not price shocks that only affect expectations about the future. Disaster risk helps the model generate the large risk premia that we observe on short-term claims.<sup>6</sup> That said, Gabaix’s (2012) model is not a complete quantitative description of financial markets; we simply view it as giving a set of

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<sup>4</sup>Also see Branger and Völkert (2010) and Zhou and Zhu (2012) for discussions. Barras and Malkhozov (2014) study the determinants of changes in the variance risk premium over time.

<sup>5</sup>Similar problems with matching term structures of Sharpe ratios in structural models have been studied in the context of claims to aggregate market dividends by van Binsbergen, Brandt, and Koijen (2012). Our results thus support and complement theirs in a novel context. See also van Binsbergen and Koijen (2015) for a recent review of the broad range of evidence on downward sloping term structures. Our paper also relates to a large literature that looks at derivative markets to learn about general equilibrium asset pricing models, for example Backus, Chernov and Martin (2011) and Martin (2014, 2015).

<sup>6</sup>An alternative possibility is that the variance market is segmented from other markets, as in, e.g., Gabaix, Krishnamurthy, and Vigneron (2007). In that case, the pricing of risks might not be integrated between the variance market and other markets. We show, however, that our results hold not only with variance swaps, but also in VIX futures and in the options market, which is large, liquid, and integrated with equity markets, making it less likely that our results are idiosyncratic.

sufficient conditions that allow a model to match the behavior of the variance swaps.

Our work is related to three main strands of the literature. First, there is the recent work in macroeconomics on the consequences of shocks to volatility, such as Bloom (2009), Bloom et al. (2014), Christiano, Motto, and Rostagno (2014), Fernandez-Villaverde et al. (2011), and Gourio (2012, 2013). We argue that if shocks to volatility are important to the macroeconomy, then investors should be willing to pay to hedge them. The lack of a risk premium on volatility news thus argues that macro models should focus on shocks to *realized* rather than *expected* volatility.

Second, we build on the consumption-based asset pricing literature that has recently focused on the pricing of volatility, including Bansal and Yaron (2004), Drechsler and Yaron (2011), Wachter (2013), and Bansal et al. (2013). We argue that the currently available set of consumption-based models with Epstein–Zin preferences are unlikely to explain the pricing of volatility claims. Andries, Eisenbach, and Schmalz (2015) analyze a model consumption-based model that matches broad features of the variance market, while van Binsbergen and Koijen (2015) discuss other recent work on related topics.

Finally, there is a large literature studying the pricing of volatility in financial markets.<sup>7</sup> Most closely related to us is a small number of recent papers with data on variance swaps with maturities from two to 24 months, including Egloff, Leippold, and Wu (2010) and Aït-Sahalia, Karaman, and Mancini (2014), who study no-arbitrage term structure models. The pricing models we estimate are less technically sophisticated than that of Aït-Sahalia, Karaman, and Mancini (2014), but we complement and advance their work in three ways. First, we examine a vast and novel range of data sources. For S&P 500 variance swaps, our panel includes data at both shorter and longer maturities than in previous studies – from one month to 14 years. The one-month maturity is important for giving a claim to shorter-term realized variance, which is what we find is actually priced. Having data at very long horizons is important for testing models, like Epstein–Zin preferences, in which expectations at very long horizons are the main drivers of asset prices. In addition, we are the first to examine the term structure of variance swaps for major international indexes, as well as for the term structure of the VIX obtained from options on those indexes. We are thus able to confirm that our results hold across a far wider range of markets, maturities, and time periods than

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<sup>7</sup>A number of papers study the pricing of volatility in options markets, e.g. Jackwerth and Rubinstein (1996), Coval and Shumway (2001), Bakshi and Kapadia (2003), Broadie, Chernov and Johannes (2009), Christofferson, Jacobs, Ornathanalai, and Wang (2008), and Kelly, Pastor, and Veronesi (2014). Lu and Zhu (2010) and Mencia and Sentana (2013) study VIX futures markets, while Bakshi, Panayotov, and Skoulakis (2011) show how to construct forward claims on variance with portfolios of options. In the Treasury bond market, Cieslak and Povala (2014) find, similar to us, that short-run volatility is more strongly priced than long-run volatility. See also Amengual and Xiu (2014) for an important recent study of jumps in volatility.

previously studied.

Our second contribution to the previous term structure literature is that rather than working exclusively within the context of a particular no-arbitrage pricing model for the term structure of variance claims, we derive from the data more general and model-independent pricing facts. Our results can be directly compared against the implications of different structural economic models, which would be more difficult if they were only derived within a specific no-arbitrage framework. Our key finding, that purely transitory realized variance is priced while innovations to longer term volatility expectations are not, can be obtained from a simple reduced-form analysis and in data both for the United States and other countries. Nevertheless, we also confirm our results in a more formal no-arbitrage setting, whose main advantage is to yield much more precise estimates of risk prices.

Our third and most important contribution is to explore the implications of variance swaps for testing structural economic models. Our theoretical analysis leads us to the conclusion that the empirical facts in the variance swap market are most consistent with a model in which variance swaps are used to hedge the realization of market crashes and in which variation in expected future stock market volatility is not priced by investors, counter to the predictions of recent asset pricing and macroeconomic models.

The remainder of the paper is organized as follows. Section 2 describes the novel datasets we obtain for variance swap prices. Section 3 reports unconditional means for variance swap prices and returns, which demonstrate our results in their simplest form. Section 4 analyzes the cross-sectional and time-series behavior of variance swap prices and returns more formally in standard asset pricing frameworks. In section 5, we discuss what structural general-equilibrium models can fit the data. We calibrate three leading models from the literature, comparing them to our data, showing that only one matches the key stylized facts. Section 6 concludes.

## 2 The data

### 2.1 Variance swaps

We focus primarily on variance swaps. Variance swaps are contracts in which one party pays a fixed amount at maturity, which we refer to as price of the variance swap, in exchange for a payment equal to the sum of squared daily log returns of the underlying asset occurring until maturity. In this paper, the underlying is the S&P 500 index unless otherwise specified.

The payment at expiration of a variance swap initiated at time  $\tau$  and with maturity  $m$  is

$$\text{Payoff } f_{\tau}^m = \sum_{j=\tau+1}^{\tau+m} r_j^2 - VS_{\tau}^m \quad (1)$$

where time here is indicated in days,  $r_j$  is the log return on the underlying on date  $j$ , and  $VS_{\tau}^m$  is the price on date  $\tau$  of an  $m$ -day variance swap. We focus on variance swaps because they give pure exposure to variance, their payoffs are transparent and easy to understand, they have a relatively long time-series, and they are relatively liquid.

Our main analysis focuses on two proprietary datasets of quoted prices for S&P 500 variance swaps.<sup>8</sup> Dataset 1 contains monthly variance swap prices for contracts expiring in 1, 2, 3, 6, 12, and 24 months, and includes data from December, 1995, to October, 2013. Dataset 2 contains data on variance swaps with expirations that are fixed in calendar time, instead of fixed maturities. Common maturities are clustered around 1, 3, and 6 months, and 1, 2, 3, 5, 10, and 14 years. Dataset 2 contains prices of contracts with maturities up to five years starting in September, 2006, and up to 14 years starting in August, 2007, and runs up to February, 2014. We apply spline interpolation to each dataset to obtain the prices of variance swaps with standardized maturities covering all months between 1 and 12 months for Dataset 1 and between 1 and 120 months for Dataset 2 (though in estimating the no-arbitrage model below we use the original price data without interpolation).

Both variance swap datasets are novel to the literature. Variance swap data with maturities up to 24 months as in Dataset 1 has been used before (Egloff, Leippold, and Wu, 2010, Ait-Sahalia, Karaman, and Mancini, 2014, and Amengual and Xiu, 2014), but the shortest maturity previous studies observed was two months. We show that the one-month variance swap is special in this market because it is the exclusive claim to next month's realized variance, which is by far the most strongly priced risk in this market. Observing the one-month variance swap is critical for precisely measuring the price of realized-variance risk.

This is also the first paper to observe and use variance swap data with maturity longer than two years. Since Epstein–Zin preferences imply that it is the very low-frequency components of volatility that should be priced (Branger and Volkert, 2010; Dew-Becker and Giglio, 2014), having claims with very long maturities is important for effectively testing the central predictions of Epstein–Zin preferences.

The variance swap market is large: the notional value of outstanding variance swaps at

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<sup>8</sup>Both datasets were obtained from industry sources. Dataset 1 is obtained from a hedge fund. Dataset 2 is obtained from Markit Totem, and reports averages of quotes obtained from dealers in the variance swap market, on average 11.

the end of 2013 was \$4 billion of notional vega.<sup>9</sup> \$4 billion of notional vega means that an increase in annualized realized volatility of one percentage point induces total payments of \$4 billion. This market is thus small relative to the aggregate stock market, but it is non-trivial economically.

We obtained information about average bid-ask spreads by maturity from a large market participant. Typical bid-ask spreads are 1 to 2 percent for maturities up to 1 year, 2 to 3 percent between 1 and 2 years, and 3 to 4 percent for maturities up to 10 years. The bid-ask spreads are thus non-trivial, but also not so large as to prohibit trading. Moreover, they are small relative to the volatility of the prices of these contracts. At the short end, the spreads are comparable to those found for corporate bonds by Bao, Pan, and Wang (2011).

Table 1 shows the total volume in notional vega terms for all transactions between March 2013 and June 2014, obtained from the DTCC (see Appendix A.1). In little more than a year, the variance swap market saw \$7.2 billion of notional vega traded. Only 11 percent of the volume was traded in short maturity contracts (1-3 months); the bulk of the transactions occurred for maturities between 6 months and 5 years, and the median maturity was 12 months.

Since these datasets are new to the literature, we devote Appendix A.1 to a battery of tests to ensure the quality of the data. In particular, we verify that: neither dataset contains stale prices (at the monthly frequency, which is the one we observe); the two datasets contain essentially the same information when they overlap (correlation above 0.997); quotes from the two datasets correspond closely to the prices for actual trades we observe since 2013; and prices in the variance swap market are extremely highly correlated with other related markets (synthetic variance swaps constructed from options as described below, and VIX futures).

In addition to the prices of S&P 500 variance swaps, we also obtained prices for variance swaps in 2013 and 2014 for the FTSE 100 (UK), Euro Stoxx 50 (Europe), and DAX (Germany) indexes. This is the first paper to examine volatility claims in international markets and we show that our main results are consistent globally.

## 2.2 Options

It is well known that variance swaps can be synthesized as a portfolio of all available out-of-the money options (Jiang and Tian (2005); Carr and Wu (2009)). The synthetic variance

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<sup>9</sup>See the Commodity Futures Trading Commission's (CFTC) weekly swap report. The values reported by the CFTC are consistent with data obtained from the Depository Trust & Clearing Corporation that we discuss below.

swap portfolio is used to construct the CBOE's VIX index. Options thus give an alternative source of information about the pricing of variance risk.

The VIX is usually reported for a 30-day maturity, but the formulas are valid at any horizon (see Appendix A.2 for details on construction). While other portfolios of options can be constructed that are also exposed to volatility in some way, the VIX is unique in that it represents a pure claim to the variance at the chosen horizon (or the squared variation when there are jumps), without any direct exposure to price movement of the underlying, independently of the horizon.<sup>10</sup>

The VIX is calculated based on an extraordinarily deep market. Options are traded in numerous venues, have notional values outstanding of trillions of dollars, and have been thoroughly studied.<sup>11</sup> Since options are exchange-traded, they involve no counterparty risk, so we can use them to check whether our results for variance swaps are affected by counterparty risk.

We construct VIX-type portfolios for the S&P 500, FTSE 100, Euro Stoxx 50, DAX, and CAC 40 indexes using data from Optionmetrics. We confirm our main results by showing that term structures and returns obtained from investments in options are similar to those obtained from variance swaps.<sup>12</sup>

## 2.3 VIX futures

Futures have been traded on the VIX since 2004. The VIX futures market is significantly smaller than the variance swap market, with current outstanding notional vega of approximately \$500 million.<sup>13</sup> Bid/ask spreads are smaller than what we observe in the variance swap market, at roughly 0.1 percent, but as the market is smaller, we would expect price impact to be larger (and market participants claim that it is). We collected data on VIX futures prices from Bloomberg since their inception and show below that they yield nearly

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<sup>10</sup>For example, an alternative strategy to obtain variance exposure is a straddle. Relative to the VIX, straddles are easier to construct, but they are claims on the absolute value of the return, not its variance, which makes the term structure more difficult to interpret given that expected absolute values are not summable across periods. In addition, obtaining exposure to realized variance through straddles requires constant rebalancing, while variance swaps require no rebalancing throughout their lives.

<sup>11</sup>Even in 1990, Vih (1990) noted that the CBOE was highly liquid and displayed little evidence of price impact for large trades.

<sup>12</sup>Recently, Boguth et al. (2012a,b) argue that returns measured on options portfolios can be substantially biased by noise, one potential source of which is the bid/ask spread. The majority of our results pertain directly to prices of volatility claims, as opposed to their returns, meaning that the issues noted by Boguth et al. are unlikely to affect our analysis. Furthermore, when we analyze returns, the portfolios are not levered to the degree that Boguth et al. argue causes biases in results.

<sup>13</sup>According to the CBOE futures exchange market statistics. See: <http://cfe.cboe.com/Data/HistoricalData.aspx>

identical results to variance swaps.

More recently, a market has developed in exchange-traded notes and funds available to retail investors that are linked to VIX futures prices. These funds currently have an aggregate notional exposure to the VIX of roughly \$5 billion, making them comparable in size to the variance swap market.

### 3 The term structure of variance claims

In this section we study average prices and returns of variance swaps. The key result that emerges is that only very short-duration variance claims earn a significant risk premium in our data. The annual Sharpe ratio for bearing transitory volatility risk is estimated to be 1.3, whereas the Sharpe ratio for exposure to shocks to *expected* future volatility 3 or more months out is estimated to be economically and statistically insignificant.

#### 3.1 Variance Swap Prices

The shortest maturity variance swap we consistently observe has a maturity of one month, so we treat a month as the fundamental period of observation. We define  $RV_t$  to be realized variance – the sum of squared daily log returns – during month  $t$ . The subscript from here forward always indexes months, rather than days.

Given a risk-neutral (pricing) measure  $Q$ , the price of an  $n$ -month variance swap at the end of month  $t$ ,  $VS_t^n$ , is

$$VS_t^n = E_t^Q \left[ \sum_{j=1}^n RV_{t+j} \right] \quad (2)$$

where  $E_t^Q$  denotes the mathematical expectation under the risk-neutral measure conditional on information available at the end of month  $t$ .

Since an  $n$ -month variance swap is a claim to the sum of realized variance over months  $t + 1$  to  $t + n$ , it is straightforward to compute prices of forward claims on realized variance. Specifically, we define an  $n$ -month variance forward as an asset with a payoff equal to realized variance in month  $t + n$ . The absence of arbitrage implies

$$F_t^n \equiv E_t^Q [RV_{t+n}] \quad (3)$$

$$= VS_t^n - VS_t^{n-1} \quad (4)$$

$F_t^n$  represents the market's risk-neutral expectation of realized variance  $n$  months in the

future (at the end of month  $t$ ). We use the natural convention that

$$F_t^0 = RV_t \tag{5}$$

so that  $F_t^0$  is the variance realized *during* the current month  $t$ . A one-month variance forward is exactly equivalent to a one-month variance swap,  $F_t^1 = VS_t^1$ .

Figure 1 plots the time series of forward variance prices for maturities between one month and ten years. The figure shows all series in annualized percentage volatility units, rather than variance units:  $100 \times \sqrt{12 \times F_t^n}$  instead of  $F_t^n$ . It also plots annualized realized volatility,  $100 \times \sqrt{12 \times F_t^0}$ , in each panel. The top panel plots forward variance claim prices for maturities below one year, while the bottom panel focuses on maturities longer than one year.

The term structure of variance claim prices is usually weakly upward sloping. In times of distress, though, such as during the financial crisis of 2008, the short end of the curve spikes, temporarily inverting the term structure. Volatility obviously was not going to continue at crisis levels, so markets priced variance swaps with the expectation that it would fall in the future.

Figure 2 reports the average term structure of forward variance claims for two different subperiods – 2008–2014, a relatively short sample for which we have data for longer maturities, is in the top panel, while the full sample, 1996–2013, is in the bottom panel. The first point on the graph (maturity 0) corresponds to the average realized volatility, whereas all points from 1 on are forward claims of different maturity.

Figure 2 shows that the term structure of forward variance claim prices is upward sloping on average, but also concave, flattening out very quickly as the maturity increases. In particular, the curve is much steeper at the very short end than everywhere else. For example, the top panel shows that the three-month claim is 30 percent more expensive than realized volatility on average, but from the three-month claim to the 120-month claim, the price rises only by another 20 percent. The bottom panel shows that the 12-month claim is only 5 percent more expensive than the three-month claim.

The average forward variance term structures in Figure 2 provide the first indication that the compensation for bearing risk associated with news about future volatility is small in this market. The return on holding a variance forward for a single month is  $\frac{F_{t+1}^n - F_t^n}{F_t^n}$ : it clearly depends on the price difference between  $n$ -maturity forwards and  $n - 1$ -maturity forwards. The average return is therefore closely related to the slope of the forward variance

term structure.<sup>14</sup>

If this curve is upward sloping between maturities  $n - 1$  and  $n$ , forward claims of maturity  $n$  will have negative average returns, implying that it is costly to buy insurance against increases in future expected volatility  $n - 1$  months ahead. The fact that the curve is very steep at short horizons and flat at long horizons is a simple way to see that it is only the claims to variance in the very near future that earn significant negative returns.

To see whether the shape of the curve is well measured statistically, figure 3 plots the average slope ( $F_t^n - F_t^{n-1}$ ) and curvature ( $(F_t^{n+1} - F_t^n) - (F_t^n - F_t^{n-1})$ ) at each maturity along with confidence intervals calculated using the Newey–West (1987) method with 6 lags. The top panel of figure 3 shows that the slopes are well identified – the slope falls from 3.7 annualized percentage points at the one-month maturity to an insignificant 0.3 percentage points at three months. The slope is also uniformly declining with maturity. The bottom panel of figure 3 plots the average curvature of the term structure. The term structure is concave on average at every maturity (and statistically significant at 7 of 11 maturities). Figure 3 thus confirms that the basic intuition from figure 2, that the term structure is steep at short maturities and essentially perfectly flat on average at longer maturities, is well measured statistically.

The top and bottom panel of Figure 2 differ in both the time period and the maturities displayed. To check the robustness of our conclusions about the average shape of the term structure of variance forwards over the period used to construct it, figure 4 examines the average term structure in different subsamples, focusing on the maturities up to 12 months to make the comparison easier. The figure shows that after 2008 the curve became slightly steeper for maturities above 1 month. However, even after 2008 the curve is still much flatter at maturities above 3 months than it is at the very short end, displaying the same pattern as in the full sample. Of course, the economic significance of the “flatness” of the curve

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<sup>14</sup>More precisely, let us first look at unscaled returns, defined as:

$$\hat{R}_{t+1}^n = F_{t+1}^{n-1} - F_t^n$$

Taking unconditional expectations of this equation we obtain:

$$E[\hat{R}_{t+1}^n] = E[F_{t+1}^{n-1}] - E[F_t^n]$$

Since the term structure is stationary,  $E[F_{t+1}^{n-1}] = E[F_t^{n-1}]$ , and therefore:

$$E[\hat{R}_{t+1}^n] = E[F_t^{n-1}] - E[F_t^n]$$

The average slope of the term structure between maturities  $n$  and  $n - 1$  corresponds to the risk premium on the  $n$ -maturity zero-coupon claim. A similar relation holds – after a loglinear approximation – for scaled returns as well: the average scaled returns at a certain maturity corresponds to the slope of the curve at that maturity relative to the level at the same maturity.

must be understood within the context of a model. In section 5 we show formally that the curve of forward variance swaps is “too flat” in both subperiods relative to the implications of workhorse asset pricing models. Finally, we also report in the figure the curve obtained excluding the peak of the financial crisis (9/2008 to 6/2009). The shape of the curve is unchanged.

### 3.2 Returns on forward variance claims

We now study the monthly returns on variance forwards. The return on an  $n$ -month variance forward corresponds to a strategy that buys the  $n$ -month forward and sells it one month later as an  $(n - 1)$ -month forward, reinvesting then again in a new  $n$ -month forward. We define the excess return of an  $n$ -period variance forward following Gorton, Hayashi, and Rouwenhorst (2013)<sup>15</sup>

$$R_{t+1}^n = \frac{F_{t+1}^{n-1} - F_t^n}{F_t^n} \quad (6)$$

Given the definition that  $F_t^0 = RV_t$ , the return on a one-month forward,  $R_{t+1}^1$  is simply the percentage return on a one-month variance swap. We focus here on the returns for maturities of one to 12 months, for which we have data since 1995. All the results extend to higher maturities in the shorter sample.

Table 2 reports descriptive statistics for our panel of monthly returns. Only the average returns for the one- and two-month maturities are negative, while all the others are weakly positive. Return volatilities are also much higher at short maturities, though the long end still displays significant variability – returns on the 12-month forward have an annual standard deviation of 17 percent, which indicates that expectations of 12-month volatility fluctuate significantly over time.

Finally, note that only very short-term returns have high skewness and kurtosis. A buyer of short-term variance swaps is therefore potentially exposed to counterparty risk if realized variance spikes and the counterparty defaults. This should induce her to pay less for the insurance, i.e. we should expect the average return to be *less negative*. Therefore, the presence of counterparty risk on the short end of the term structure would bias our estimate towards not finding the large negative expected returns that we instead find. On the other hand, returns at longer maturities have much lower skewness and kurtosis, which indicates

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<sup>15</sup>Note that  $F_{t+1}^{n-1} - F_t^n$  is also an excess return on a portfolio since no money changes hands at the inception of a variance swap contract. Following Gorton, Hayashi, and Rouwenhorst (2013), we scale the return by the price of the variance claim bought. This is the natural scaling if the amount of risk scales proportionally with the price, as in Cox, Ingersoll, and Ross (1985). We have reproduced all of our analysis using the unscaled excess return  $F_{t+1}^{n-1} - F_t^n$  as well and confirmed that all the results hold in that case.

that counterparty risk is substantially less relevant. Finally, we note that we obtain the same results below using options, which are exchange traded and not affected by counterparty risk.

Given the different volatilities of the returns at different ends of the term structure, it is more informative to examine Sharpe ratios, which measure compensation earned per unit of risk. Figure 5 shows the annual Sharpe ratios of the 12 forwards. The Sharpe ratios are negative for the one- and two-month maturities (at around -1.3 and -0.4, respectively), but all other Sharpe ratios are insignificantly different from zero, and in fact slightly positive. Statistically, not only we don't reject the hypothesis that these Sharpe ratios are different from zero: we can statistically reject the hypothesis that the Sharpe ratios are meaningfully negative at all maturities above 3; for example, we can reject at the 95% level that the annual Sharpe ratio on a 12-month claim is below -0.11.<sup>16</sup>

Despite the relatively short sample, there is also a strongly statistically significant difference across the Sharpe ratios at the very short end of the curve and everywhere else. The annual Sharpe ratio of the 1-month variance claim is more negative by 0.9 than the 2-month claim (the p-value for the difference is 0.03), and at least 1.3 lower than the Sharpe ratio at all higher horizons (the p-values of the differences are all less than 0.01). These are enormous differences, considering for example that the annual Sharpe ratio of the aggregate stock market has historically been approximately 0.3. The one-month forward therefore yields a Sharpe ratio statistically and economically much more negative than any other forward that we observe.

Any claim to volatility at a horizon beyond one month is purely exposed to news about future volatility: its return corresponds exactly to the change in expectations about volatility at its maturity. Pure news about future expected volatility will therefore affect its return, whereas purely transitory shocks to volatility that disappear before its maturity will not affect it at all. Our results therefore show that news about future volatility commands a small to zero risk premium in our data.

The results at the short end of the curve indicate that investors are willing to pay a large premium to hedge realized volatility. What is new and surprising in this picture is the fact that investors are willing to pay much less (economically and statistically zero) to hedge any

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<sup>16</sup>One may also worry that some of our results depend on the interpolation between observed maturities. To make sure this does not affect our results, we have constructed 6-month holding period returns of a claim to variance 6 to 12 months forward (which we refer to as the 6/12 portfolio) as well as the 3-month return of a claim to variance 3 to 6 months forward (the 3/6 portfolio). None of these returns depend on interpolated data. The annualized Sharpe ratios on these two portfolios are statistically indistinguishable from 0, and economically small and in fact slightly positive (0.11 for the 6/12 portfolio, with standard error 0.20, and 0.03 for the 3/6 portfolio, standard error 0.22), consistent with the results in the figure. Of course, the return of the 1-month claim (Sharpe ratio of -1.3 as reported in the figure) also does not depend on interpolated data.

innovations in *expected* volatility. The estimated Sharpe ratio is zero (or even positive) at *every* point in the curve above maturity 3 months. Moreover, these declining Sharpe ratios are consistent with the findings of van Binsbergen and Koijen (2015), who find that Sharpe ratios in a range of markets decline with maturity.<sup>17</sup> Like them, we show below that our results are difficult to reconcile with standard theories, thus further extending the puzzle originally set forth by van Binsbergen, Brandt, and Koijen (2012).

### 3.3 Evidence from other markets

Figure 6 shows the term structure of prices and Sharpe ratios of variance forwards obtained from the variance swap data compared to the synthetic claims for maturities up to 1 year. While the curves obtained using options data seem noisier, the curves deliver the same message: the volatility term structure is extremely steep at the very short end but quickly flattens out for maturities above two months, and Sharpe ratios rapidly approach zero as the maturity passes two months.<sup>18</sup> Appendix Figure A.3 shows that we obtain similar results with VIX futures.<sup>19,20</sup>

Our results also extend to international markets. Figure 7 plots average term structures obtained from both variance swaps and synthetic option-based variance claims for the Euro Stoxx 50, FTSE 100, CAC 40 and DAX indexes. Both panels of the figure show that the international term structures have an average shape that closely resembles the one observed for the US (the solid line in both panels), demonstrating that our results using US variance swaps extend to the international markets.<sup>21</sup>

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<sup>17</sup>The declining term structure of Sharpe ratios on short positions in volatility is consistent with the finding of van Binsbergen, Brandt, and Koijen (2012) that Sharpe ratios on claims to dividends decline with maturity, and that of Duffee (2011) that Sharpe ratios on Treasury bonds decline with maturity. For a review, see van Binsbergen and Koijen (2015).

<sup>18</sup>Given the high liquidity of the options market, we might have expected option-based portfolios to be less noisy. However, the synthetic variance portfolios load heavily on options very far out of the money where liquidity is relatively low. This demonstrates another advantage of studying variance swaps instead of options.

<sup>19</sup>VIX futures are not exactly comparable to variance swaps because they are claims on the  $VIX$ , not on  $VIX^2$ . A convexity effect makes the prices of claims on variance and volatility different, but the Figure shows that it is quantitatively small.

<sup>20</sup>We have also compared our data to the CBOE's 3-month and mid-term volatility indexes (VXV and VXMT) and find that our data is nearly identical to those two series in the period when the CBOE calculates them.

<sup>21</sup>In the appendix (Figure A.2) we also confirm that for the indexes for which we have both variance swap prices and synthetic prices obtained from options, the two curves align well.

## 4 Asset pricing

### 4.1 Reduced-form estimates

We begin by exhibiting our main pricing result in a simple reduced-form setting: investors pay to hedge realized volatility but not shocks to expected volatility.

#### 4.1.1 Extracting innovations

As usual in the term structure literature, we begin by extracting principal components from the term structure of variance forwards. The first factor explains 97.1 percent of the variation in the term structure and the second explains an additional 2.7 percent. The loadings of the variance swaps on the factors are plotted in the top panel of Figure 8, while the time series of the factors are shown in the bottom panel of the figure. The first factor captures the level of the term structure, while the second measures the slope. As we would expect, during times of crisis, the slope turns negative. The level factor captures the longer-term trend in volatility and clearly reverts to its mean more slowly.

To extract shocks to variance and expectations, we estimate a first-order vector autoregression (VAR) with the two principal components and realized variance ( $RV$ ). Including  $RV$  in the VAR allows us to separately identify shocks to the term structure of variance swaps and transitory shocks to realized variance.

We rotate the three shocks using a Cholesky factorization where the first shock affects all three variables, the second affects only the slope and  $RV$ , and the third shock affects only  $RV$ . We will therefore refer to the third shock as the pure  $RV$  shock. The pure  $RV$  shock allows us to measure the price of risk for a shock that has only a transitory effect on realized variance and no effect on the term structure of variance swap prices, while the other two rotated shocks affect both current realized variance and also expectations of future variance. The factorization also normalizes all three shocks to have unit variance. Impulse response functions are reported in Appendix Figure A.4.

#### 4.1.2 Risk prices

We estimate risk prices for the three shocks using the Fama–MacBeth (1973) procedure on 1- to 12-month variance forwards.<sup>22</sup> The top panel of Table 3 reports the loadings of each forward return on the three orthogonalized shocks. Short-maturity forwards are exposed to

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<sup>22</sup>The results are robust to estimating the risk prices using one- and two-step GMM.

all three shocks with the expected signs. The higher maturities are mostly exposed to the level and slope shocks, with essentially no exposure to the pure  $RV$  shock.

The bottom panel of Table 3 reports the estimated annualized risk prices. Of the three shocks, only the pure  $RV$  shock has a statistically significant risk price. The risk price is also economically highly significant: it implies that an asset that was exposed only to the pure  $RV$  shock would earn an annualized Sharpe ratio of -2.69. Since the three shocks all have the same standard deviation, the magnitudes of the risk prices are directly comparable. Those for shocks 1 and 2 are four to six times smaller than that for the pure  $RV$  shock, and thus economically far less important. Statistically, we can reject the hypothesis that the price of risk for  $RV$  is the same as that for either shock 1 or 2 at the one-percent level.<sup>23</sup>

Table 3 thus shows that investors do not price shocks to the level and slope, but they accept large negative returns to hedge transitory  $RV$  shocks. Since the level and slope factors explain 99.9 percent of the variation in variance swap prices, they encode essentially all the information in the term structure. The fact that the shocks to those two factors are unpriced therefore implies that no forward-looking information about volatility that appears in asset prices is significantly priced.

### 4.1.3 Controlling for the market return

One possible explanation for why realized variance is priced is that it provides a good hedge for aggregate market shocks. To test that possibility, we add the market return as an additional factor in the estimation.<sup>24</sup> The first column of Table 4 shows that indeed the forward volatility claims are heavily exposed to the market return. But when the pure  $RV$  shock is included, the market return is no longer significantly priced. The  $R^2$  of the model for the cross-section of average returns also rises from 38.3 to 99.8 percent when the pure  $RV$  shock is included.

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<sup>23</sup>Despite the good fit of the model in terms of  $R^2$ , the GMM and the GRS test reject the null that all the average pricing errors are zero. This is because the pricing errors, while being small relative to the overall average returns of these contracts, are still statistically different from zero. The same applies to all cross-sectional tests in the next sections.

<sup>24</sup>We add the market return as a test asset to impose discipline on its risk premium. For readability and to ensure that the risk premium on the market is matched relatively closely, we increase the weight on the market return by a factor of 12 as a test asset in our cross-sectional tests. That way, the market return carries as much weight in the pricing tests as do all the variance claims combined. The market factor, though, is still the monthly market return, as are all our forward variance returns.

#### 4.1.4 Upside and downside volatility

A natural question is whether investors desire to hedge all volatility shocks, or whether they primarily desire to hedge volatility during downturns. Segal, Shaliastovich, and Yaron (2015), for example, discuss such a model. Following Andersen and Bondarenko (2007), we decompose the realized variance in a month,  $RV_t$ , into an upper and a lower semivariance: the integrated realized variances computed only when prices are above or below a threshold. In particular, following Andersen and Bondarenko (2007) we construct the upper  $RV$  in each month as

$$RV_t^U = \sum_{j \in t} (r_j)^2 I_j(P_j > P_0)$$

where  $j \in t$  indicates days  $j$  in month  $t$ ,  $I_j(P_j > P_0)$  is an indicator that the futures price of the index  $P_j$  of the underlying in day  $j$  is above the starting point  $P_0$  at the beginning of the month. Similarly, we construct

$$RV_t^D = \sum_{j \in t} (r_j)^2 I_j(P_j \leq P_0)$$

Andersen and Bondarenko (2007) discuss two useful properties of these realized barrier variances (or semivariances), which can be interpreted as the volatility of the upward and downward price movements. First, the two components sum to  $RV_t$ ,

$$RV_t = RV_t^U + RV_t^D$$

Second, the price of claims to  $RV_t^U$  and  $RV_t^D$  can be obtained from option prices in a manner similar to how the  $VIX$  is computed. We refer to these two prices as  $VIX_t^U$  and  $VIX_t^D$ .  $(VIX_t^U)^2$  is the no-arbitrage price of a contract whose payoff is  $RV_{t+1}^U$ , and  $(VIX_t^D)^2$  is the no-arbitrage price of a contract whose payoff is  $RV_{t+1}^D$ . Andersen and Bondarenko (2007) also derive a relation between the three prices,

$$VIX_t^2 = (VIX_t^U)^2 + (VIX_t^D)^2$$

Finally, just as in the case of the  $VIX$ , we can compute the prices of the two claims for different maturities and study the term structure.

Figure 9 plots the term structure of the variance forwards obtained from  $VIX$ , as well as those for  $VIX^U$  and  $VIX^D$ . As before, maturity zero corresponds to the average  $RV_t$ ,  $RV_t^U$  and  $RV_t^D$ , respectively. The slopes between the zero- and one-month maturities then represent precisely the returns on the 30-day  $VIX$ ,  $VIX^U$ , and  $VIX^D$ . We can see that

most of the negative average return that investors are willing to accept to hold the VIX comes from the extremely negative monthly return of the  $VIX^D$  (about -30% per month), while  $VIX^U$  commands a return much closer to zero.<sup>25</sup> This confirms the intuition that the reason investors hedge realized volatility is due to its downside component (which Bollerslev and Todorov (2011) show is dominated by downward jumps).

## 4.2 The predictability of volatility

Since the key result of the paper concerns the pricing of volatility shocks at different horizons, a natural question is how much news there actually is about future volatility. The total risk premium for assets that hedge volatility news – which we showed at the beginning of Section 3 to be insignificantly different from zero – is the product of the quantity of risk (how much news about future volatility investors receive), and the price of this risk (how averse investors are to such news). Perhaps the reason that investors are willing to pay little to hedge volatility news is that the quantity of news is small. We show here that in fact investors do receive news about future volatility and that they seem indifferent to that news.

First, note that the results in the previous section *already* explicitly focus on the *price* of risk of volatility, rather than the *quantity*. The reported risk prices measure compensation per unit of risk, so they are unaffected by how much news there is about future volatility. If volatility were not very predictable, the quantity of news risk would be low, but the price per unit of risk would still be estimated correctly from our cross-sectional regressions. So our previous analysis already shows that the low risk premia are due to a low price of variance news risk.<sup>26</sup>

It is also useful to remember that the sample mean Sharpe ratios are insignificantly different from zero (and in fact positive) even for maturities as short as three months. But there is very strong evidence in the literature that volatility is predictable three months ahead – so that the result of zero risk premium cannot stem from zero quantity of news risk at that horizon (see for example Andersen et al. (2003)). Indeed, the volatility literature has demonstrated predictability at horizons much longer than three months.<sup>27</sup>

To quantify the magnitude of the predictability of volatility at different horizons Table

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<sup>25</sup>Note that contrary to the case of the VIX, for  $VIX^U$  and  $VIX^D$  the slope between maturities above one month cannot be interpreted exactly in terms of returns since the barrier is moving over time.

<sup>26</sup>Of course, the lower the amount of risk, the harder it is to estimate the price of risk; this effect is fully captured by the standard errors on the estimates of the price of risk.

<sup>27</sup>Andersen et al. (2003), Ait-Sahalia and Mancini (2008), Bandi, Russell, and Yang (2008), and Brownlees, Engle, and Kelly (2011) show that volatility is predictable based on lagged returns of the underlying and past volatility. Campbell et al. (2014) focus on longer horizons (up to 10 years) and show that both the aggregate price-earnings ratio and the Baa-Aaa default spread are useful predictors of long-run volatility.

5 reports  $R^2$ s from predictive regressions for realized volatility at different frequencies and horizons. The first pair of columns focuses on forecasts of monthly realized variance, while the second pair repeats the exercise at the annual frequency. The  $R^2$ s for monthly volatility range from 45 percent at the 1-month horizon to 20 percent at the 12-month horizon. In predicting annual volatility,  $R^2$ s range between 56 and 21 percent for horizons of 1 to 10 years.

The third pair of columns in Table 5 reports, as a comparison, the results of forecasts of dividend growth.<sup>28</sup>  $R^2$ s for dividend growth are never higher than 9 percent. So in the context of financial markets, there is an economically large amount of predictability of volatility. The appendix takes an extra step beyond Table 5 and shows, using Fama and Bliss (1987) and Campbell and Shiller (1991) regressions, that nearly all the variation in variance swap prices is actually due to variations in expected volatility, rather than risk premia.

We conclude by noting that while there is ample evidence of the predictability of volatility at the horizons relevant for this analysis (from 3 months upwards), the result that the risk premium for volatility news is close to zero would have strong implications for macroeconomic and financial models even if it was driven by low quantity of expected volatility risk. If there is not much volatility news, then asset pricing models in which news about future volatility plays an important role (like the ICAPM or several versions of the long-run-risks model) would lose this source of priced risk; similarly the macro literature showing that volatility news can drive the business cycle would seem irrelevant if there is no volatility news.

### 4.3 A no-arbitrage model

In this section, we extend the pricing results reported above by considering a more formal estimation. We analyze a standard no-arbitrage term structure model for variance swaps. The model delivers implications strongly supportive of our reduced-form results. Because the no-arbitrage model uses the prices of the variance swaps, rather than just their returns, and because it uses a full no-arbitrage structure, it is able to obtain much more precise estimates of risk prices. We show that not only are the risk prices on the level and slope factors statistically insignificant, but they are also economically small.

The no-arbitrage model has three additional advantages over the reduced-form analysis: it explicitly allows for time-variation in the volatility of shocks to the economy and risk prices, the standard errors for the risk prices take into account uncertainty about the dynamics of the economy (through the VAR), and it links us more directly to the previous literature.

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<sup>28</sup>We compare predictability of volatility to that of dividends since realized variance in each month is the stochastic payment of the variance swap contract in that month.

Furthermore, because the inputs to the estimation of the no-arbitrage model are the observed variance swap prices rather than monthly returns, the results in this section do not rely on any interpolation and we can simultaneously use the full time series from 1996 to 2013 and every maturity from one month to 14 years.

### 4.3.1 Risk-neutral dynamics

As above, we assume that the term structure of variance swaps is governed by a bivariate state vector  $(s_t^2, l_t^2)'$ . Rather than state the factors as a level and slope, we now treat them as a short- and a long-term component, which will aid in the estimation process.  $s_t^2$  is the one-month variance swap price:  $s_t^2 = E_t^Q [RV_{t+1}]$ . The other state variable,  $l_t^2$ , governs the central tendency of  $s_t^2$ .

We begin by specifying the conditional risk-neutral mean of the states,

$$E_t^Q \left[ \begin{pmatrix} s_{t+1}^2 \\ l_{t+1}^2 \\ RV_{t+1} \end{pmatrix} \right] = \begin{pmatrix} \rho_s^Q & 1 - \rho_s^Q & 0 \\ 0 & \rho_l^Q & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} s_t^2 \\ l_t^2 \\ RV_t \end{pmatrix} + \begin{pmatrix} 0 \\ v_l^Q \\ 0 \end{pmatrix} \quad (7)$$

where  $v_l^Q$  is a constant to be estimated which captures the unconditional mean of realized variance.  $l_t^2$  can be viewed as the risk-neutral trend of  $s_t^2$ . The first two rows of (7) are the discrete-time counterpart to the standard continuous-time setup in the literature, e.g. Egloff, Leippold, and Wu (2010) and Ait-Sahalia, Karaman, and Mancini (2014).<sup>29</sup> We diverge from Egloff, Leippold, and Wu (2010) and Ait-Sahalia, Karaman, and Mancini (2014) in explicitly specifying a separate process for realized variance, noting that it is not spanned by the other shocks. The specification of a separate shock to  $RV_{t+1}$  allows us to ask how shocks to both realized variance and the term structure factors are priced.<sup>30</sup>

Given the assumption that  $s_t^2 = E_t^Q [RV_{t+1}]$ , the price of an  $n$ -period variance swap  $VS_t^n$  is

$$VS_t^n = E_t^Q \left[ \sum_{i=1}^n RV_{t+i} \right] = E_t^Q \left[ \sum_{i=1}^n s_{t+i-1}^2 \right] \quad (8)$$

<sup>29</sup>For admissibility, we require that  $0 < \rho_s^Q < 1$ ,  $\rho_l^Q > 0$ , and  $v_l^Q > 0$ . These restrictions ensure that risk-neutral forecasts of  $s_t^2$  and  $l_t^2$ , hence variance swap prices at various maturities, are strictly positive.

<sup>30</sup>From a continuous-time perspective, it is not completely obvious how to think about a "shock" to realized variance that is completely transitory. There are two standard interpretations. One is that the innovation in  $RV_{t+1}$  represents the occurrence of jumps in the S&P 500 price. Alternatively, there could be a component of the volatility of the diffusive component of the index that has shocks that last less than one month. At some point, the practical difference between a jump and an extremely short-lived change in diffusive volatility is not obvious. The key feature of the specification is simply that there are shocks to the payout of variance swaps that are orthogonal to both past and future information contained in the term structure.

which can be computed by applying (7) repeatedly, and which implies that  $VS_t^n$  is affine in  $s_t^2$  and  $l_t^2$  for any maturity.

### 4.3.2 Physical dynamics and risk prices

Define  $X_t \equiv (s_t^2, l_t^2, RV_t)'$ . We assume that  $X$  follows a VAR(1) under the physical measure:<sup>31</sup>

$$\begin{pmatrix} s_{t+1}^2 \\ l_{t+1}^2 \\ RV_{t+1} \end{pmatrix} = \begin{pmatrix} 0 \\ v_l^Q \\ 0 \end{pmatrix} + \begin{pmatrix} \rho_s & 1 - \rho_s^Q & 0 \\ 0 & \rho_l & 0 \\ \rho_{s,RV} & 0 & 0 \end{pmatrix} \begin{pmatrix} s_t^2 \\ l_t^2 \\ RV_t \end{pmatrix} + \varepsilon_{t+1} \quad (9)$$

$$\varepsilon_{t+1} \sim N \left( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, V_t(X_{t+1}) \right) \quad (10)$$

In our main results, we follow Egloff, Leippold, and Wu (2010) and Ait-Sahalia, Karaman, and Mancini (2014) and assume that the market prices of risk are proportional to the states, so that the log SDF,  $m_{t+1}$ , is

$$m_{t+1} - E_t[m_{t+1}] = \Lambda_t' V_t(X_{t+1})^{-1/2} \varepsilon_{t+1} \quad (11)$$

$$\text{where } \Lambda_t = \begin{pmatrix} \lambda_s s_t \\ \lambda_l l_t \\ \lambda_{RV} s_t \end{pmatrix} \quad (12)$$

where the superscript  $^{1/2}$  indicates a lower triangular Cholesky decomposition. The term  $V_t(X_{t+1})^{-1/2}$  standardizes and orthogonalizes the shocks  $\varepsilon_{t+1}$ .  $\Lambda_t$  thus represents the price of exposure to a unit standard deviation shock to each component of  $X_{t+1}$ .

To maintain the affine structure of the model, we need the product  $V_t(X_{t+1})^{1/2} \Lambda_t$  to be affine in  $X_t$ . The specification for  $\Lambda_t$  in (12) is therefore typically accompanied by a structure for the conditional variance similar to that of Cox, Ingersoll, and Ross (1985),

$$V_t(X_{t+1}) = \begin{pmatrix} \sigma_s^2 s_t^2 & 0 & \sigma_{s,RV} s_t^2 \\ 0 & \sigma_l^2 l_t^2 & 0 \\ \sigma_{s,RV} s_t^2 & 0 & \sigma_{RV}^2 s_t^2 \end{pmatrix} \quad (13)$$

which guarantees that  $V_t(X_{t+1})^{1/2} \Lambda_t$  is affine in  $X_t$ .<sup>32</sup>

<sup>31</sup>Admissibility requires that  $v_l^Q$  and the feedback matrix in (9) be non-negative, which ensures that the forecasts of  $X_t$ , and hence future volatility, be strictly positive.

<sup>32</sup>It is important to note that the specifications of  $\Lambda_t$  in (12) and  $V_t(X_{t+1})$  in (13) introduce tight re-

### 4.3.3 Empirical results

The estimation uses standard likelihood-based methods. The appendix describes the details. We use both Dataset 1 and Dataset 2, meaning that the number of variance swap prices used in the estimation varies over time depending on availability.

**Model fit** Table 6 reports the means and standard deviations of the variance swap prices observed and fitted by our model together with the corresponding root mean squared errors (RMSE). The average RMSE across maturities up to 24 months is 0.73 annualized volatility points (i.e. the units in Figure 1).<sup>33</sup> For maturities longer than 24 months, since we do not have time series of variance swap prices with fixed maturities for the entire sample, we cannot report the sample and fitted moments for any fixed maturity. Instead, we stack all contracts with more than 24 months to maturity into one single series and compute the RMSE from the observed and fitted values of this series. The corresponding RMSE is reported in the last row of Table 6. At 0.87 percentage points, it compares favorably with the RMSE for the shorter maturities. Table 6 suggests that our models with two term structure factors plus  $RV$  are capable of pricing the cross-section of variance swap prices for an extended range of maturities. Even when maturities as long as 14 years are included in estimation, the data does not seem to call for extra pricing factors.

**Risk prices** The steady-state risk prices in the model are reported in Table 7 along with their standard errors. As in the previous analysis, we find clearly that it is the purely transitory shock to realized variance that is priced ( $RV$ -risk). The Sharpe ratio associated with an investment exposed purely to the transitory  $RV$  shock – analogous to the pure  $RV$  shock above – is -1.70.

In the VAR analysis in the previous section, the pure  $RV$  shock had no immediate impact on the level and slope factors, but it could potentially indirectly affect future expected variance through the VAR feedback. In the no-arbitrage model, that effect is shut off through the specification of the dynamics. That is, the  $RV$  shock here is completely transitory – it has no impact on expectations of volatility on any future date. The other two shocks are

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restrictions on the difference  $E_t(X_{t+1}) - E_t^Q(X_{t+1})$ . In the appendix, we therefore consider two alternative specifications for the variance process  $V_t(X_{t+1})$  and the risk prices  $\Lambda_t$  that are more flexible in certain dimensions. The results, both in terms of point estimates and standard errors, are essentially identical across the various specifications that we consider, so we report results for this specification here since it is most common in the literature.

<sup>33</sup>When we exclude the financial crisis, using a sample similar to that of Egloff, Leippold, and Wu (2010), we obtain an RMSE of 0.33 percentage points, which is nearly identical to their reported value. The increase in fitting error in the full sample is, not surprisingly, brought about by the large volatility spikes that occurred during the crisis.

forced to account for all variation in expectations. The fact that the results are consistent between the no-arbitrage model and the reduced-form analysis in the previous section helps underscore the robustness of our findings to different modeling assumptions.

The short- and long-term factors earn risk premia of only -0.11 and -0.18, respectively, neither of which is significantly different from zero. The lack of statistical significance is not due to particularly large standard errors; the standard errors for the risk prices for the  $s_t^2$  and  $l_t^2$  shocks are in fact substantially smaller than that for the  $RV$  shock. Moreover, Sharpe ratios of -0.11 and -0.18 are also economically small. For comparison, the Sharpe ratio of the aggregate stock market in the 1996–2013 period is 0.43. So the risk premia on the short- and long-term components of volatility are between 25 and 42 percent of the magnitude of the Sharpe ratio on the aggregate stock market. On the other hand, the Sharpe ratio for the  $RV$  shock is nearly four times larger than that for the aggregate stock market and 10 to 15 times larger than the risk prices on the other two shocks. Our no-arbitrage model thus clearly confirms the results from the previous sections.

**Time-series dynamics** The estimated parameters determining the dynamics of the state variables under the physical measure are (equation 9):

$$\begin{pmatrix} s_{t+1}^2 \\ l_{t+1}^2 \\ RV_{t+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0.99^{***} \\ 0 \end{pmatrix} + \begin{pmatrix} 0.82^{***} & 0.16^{**} & 0 \\ 0 & 0.98^{***} & 0 \\ 0.75^{***} & 0 & 0 \end{pmatrix} \begin{pmatrix} s_t^2 \\ l_t^2 \\ RV_t \end{pmatrix} + \varepsilon_{t+1}$$

The key parameter to focus on is the persistence of  $l_t^2$ . The point estimate is 0.9814, with a standard error of 0.0013. At the point estimate, long-term shocks to variance have a half-life of 37 months. That level of persistence is actually *higher* than the persistence of consumption growth shocks in Bansal and Yaron’s (2004) long-run risk model, and only slightly smaller than the persistence they calibrate for volatility, 0.987. Our empirical model thus allows us to estimate risk prices on exactly the type of long-run shocks that have been considered in calibrations. As we discuss further below, the fact that we find that the long-term shock to volatility is unpriced is strongly at odds with standard calibrations of Epstein–Zin preferences where agents are strongly averse to *news* about future volatility.

## 5 Economic interpretation

The key message of our empirical analysis is that it was costless in our sample to hedge news about future volatility, at horizons 3 months to 14 years. That result can be seen

by simply inspecting the average term structure of forward variance prices, by calculating Sharpe ratios on variance forwards, by estimating risk prices using Fama–MacBeth, and in a no-arbitrage model of the term structure. The insignificant risk price on volatility news immediately suggests that models based on standard calibrations of Epstein–Zin (1991) preferences, where intertemporal hedging effects are central, will struggle to match the data.

To confirm that intuition, we simulate two models with Epstein–Zin preferences. The first is the long-run risk model proposed by Drechsler and Yaron (2011), and the second is a discrete-time version of the model with time-varying disaster risk proposed by Wachter (2013). In both cases, we show that the models imply that the Sharpe ratios earned from rolling over long-term forward variance claims are almost as negative as those earned from holding just the one-month variance swap, counter to what we observe empirically in Figure 5. Furthermore, the models are unable to match the shape of the average term structure, in both cases being much less concave than we observe in the data.

The lack of intertemporal hedging in the variance swap market suggests a myopic model of investors. We therefore consider a simple model – based on Gabaix’s (2012) rare disaster framework – in which investors have power utility and show that it is able to match the features of the variance swap market that we have documented.<sup>34</sup>

## 5.1 Structural models of the variance premium

### 5.1.1 A long-run risk model

Drechsler and Yaron (2011), henceforth DY, extend Bansal and Yaron’s (2004) long-run risk model to allow for jumps in both the consumption growth rate and volatility. DY show that the model can match the mean, volatility, skewness, and kurtosis of consumption growth and stock market returns, and generates a large 1-month variance risk premium that forecasts market returns, as in the data. DY is thus a key quantitative benchmark in the literature.

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<sup>34</sup>We do not explicitly consider here habit formation models as in Campbell and Cochrane (1999) because conditional on the habit, the agent in that model behaves as a standard power utility investor. The habit simply shifts the level of risk aversion over time. It does not cause investors to be averse to shocks to volatility.

The power utility model that we examine is not necessarily the only model that can possibly explain the data. Andries, Eisenbach, and Schmalz (2015) consider a model with highly exotic preferences and discuss some evidence that it may be able to match our findings. Van Binsbergen and Koijen (2015) discuss other recent theoretical developments.

The structure of the endowment process is

$$\Delta c_t = \mu_{\Delta c} + x_{t-1} + \varepsilon_{c,t} \quad (14)$$

$$x_t = \mu_x + \rho_x x_{t-1} + \varepsilon_{x,t} + J_{x,t} \quad (15)$$

$$\bar{\sigma}_t^2 = \mu_{\bar{\sigma}} + \rho_{\bar{\sigma}} \bar{\sigma}_{t-1}^2 + \varepsilon_{\bar{\sigma},t} \quad (16)$$

$$\sigma_t^2 = \mu_{\sigma} + (1 - \rho_{\sigma}) \bar{\sigma}_{t-1}^2 + \rho_{\sigma} \sigma_{t-1}^2 + \varepsilon_{\sigma,t} + J_{\sigma,t} \quad (17)$$

where  $\Delta c_t$  is log consumption growth, the shocks  $\varepsilon$  are mean-zero and normally distributed, and the shocks  $J$  are jump shocks.  $\sigma_t^2$  controls both the variance of the normally distributed shocks and also the intensity of the jump shocks. There are two persistent processes,  $x_t$  and  $\bar{\sigma}_t^2$ , which induce potentially long-lived shocks to consumption growth and volatility. We follow DY's calibration for the endowment process exactly.

Aggregate dividends are modeled as

$$\Delta d_t = \mu_d + \phi x_{t-1} + \varepsilon_{d,t} \quad (18)$$

Dividends are exposed to the persistent but not the transitory part of consumption growth. Equity is a claim on the dividend stream, and we treat variance claims as paying the realized variance of the return on equities.

DY combine that endowment process with Epstein–Zin preferences, and we follow their calibration. Because there are many parameters to calibrate, we refer the reader to DY for the full details. However, the parameters determining the volatility dynamics are obviously critical to our analysis. Note that the structure of equations (16) and (17) is the same as the VAR in our no-arbitrage model in equation (9). The parameters governing volatility in DY's calibration and the corresponding values from our estimation are:

	DY	Estimates
$\rho_{\sigma}$	0.87	0.82
$\rho_{\bar{\sigma}}$	0.987	0.9814
$\text{stdev}(\varepsilon_{\bar{\sigma},t})$	0.10	0.05
$\text{stdev}(\varepsilon_{\sigma,t} + J_{\sigma,t})$	1.10	1.48

The two feedback coefficients,  $\rho_{\sigma}$  and  $\rho_{\bar{\sigma}}$ , are nearly identical to our estimated values. Their long-term component,  $\bar{\sigma}^2$ , has a persistence of 0.987, which compares favorably with our estimate of 0.9814. Similarly, their calibration of  $\rho_{\sigma} = 0.87$  is comparable to our estimate of 0.82. The calibration deviates somewhat more in the standard deviations of the innovations.

Overall, though, DY’s calibration implies volatility dynamics highly similar to what we observe empirically – so we keep it exactly as in the original paper. The close match is not surprising as DY’s model was calibrated to fit the behavior of the one-month VIX and realized variance.

Given the high quality of DY’s calibration, if the long-run risk model fails to match the term structure of variance swap prices, it is not because it has an unreasonable description of the dynamics of volatility. Rather, we would conclude that the failure is due to the specification of the preferences, namely Epstein–Zin with a representative agent.

### 5.1.2 Time-varying disaster risk

The second model we study is a discrete-time version of Wachter’s (2013) model of time-varying disaster risk. In this case, consumption growth follows the process,

$$\Delta c_t = \mu_{\Delta c} + \sigma_{\Delta c} \varepsilon_{\Delta c,t} + J_{\Delta c,t} \tag{19}$$

where  $\varepsilon_{\Delta c,t}$  is a mean-zero normally distributed shock and  $J_t$  is a disaster shock. The probability of a disaster in any period is  $F_t$ , which follows the process

$$F_t = (1 - \rho_F) \mu_F + \rho_F F_{t-1} + \sigma_F \sqrt{F_{t-1}} \varepsilon_{F,t} \tag{20}$$

The CIR process ensures that the probability of a disaster is always positive in the continuous-time limit, though it can generate negative values in discrete time. We calibrate the model similarly to Wachter (2013) and Barro (2006). Details of the calibration are reported in the appendix. The model is calibrated at the monthly frequency. In the calibration, the steady-state annual disaster probability is 1.7 percent as in Wachter (2013).  $\sigma_F$  is set to 0.0075 ( $\varepsilon_F$  is a standard normal), and  $\rho_F = 0.87^{1/12}$ , which helps generate realistically volatile stock returns and a persistence for the price/dividend ratio that matches the data. If there is no disaster in period  $t$ ,  $J_t = 0$ . Conditional on a disaster occurring,  $J_t \sim N(-0.30, 0.15^2)$ , as in Barro (2006). Finally, dividends are a claim to aggregate consumption with a leverage ratio of 2.8.<sup>35</sup>

Wachter (2013) combines this specification of disasters with Epstein–Zin preferences.

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<sup>35</sup>The occurrence of a disaster shock implies that equity values decline instantaneously. To calculate realized variance for periods in which a disaster occurs, we assume that the shock occurs over several days with maximum daily return of -5 percent. For example, a jump of 20% would occur over 4 consecutive days, with a 5% decline per day. This allows for a slightly delayed diffusion of information and also potentially realistic factors such as exchange circuitbreakers. The small shocks  $\varepsilon_{\Delta c,t}$  are treated as though they occur diffusively over the month, as in Drechsler and Yaron (2011).

One of her key results is that a model with time-varying disaster risk and power utility has strongly counterfactual predictions for the behavior of interest rates and other asset prices. She thus argues that time-varying disaster risk should be studied in the context of Epstein–Zin preferences. We follow her in assuming the elasticity of intertemporal substitution is 1. To help the model match data for the variance claim returns, which the paper did not originally targeted, we set risk aversion as high as possible to try to generate a large enough 1-month variance risk premium (which has been the focus of most previous literature on the pricing of variance risk). We therefore set it to 3.6, as the model does not have a solution when risk aversion is higher.<sup>36</sup>

### 5.1.3 Time-varying recovery rates

The final model we study is a version of Gabaix’s (2012) model of disasters with time-varying recovery rates. Because the probability of a disaster is constant, power utility and Epstein-Zin are equivalent in terms of their implications for risk premia. We use power utility in our calibration, which eliminates the intertemporal hedging motives present in the two previous models. In this model, the expected value of firms following a disaster is variable. Specifically, we model the consumption process identically to equation (19) above, but with the probability of a disaster,  $F_t$ , fixed at 1 percent per year (Gabaix’s calibration). Following Gabaix, dividend growth is

$$\Delta d_t = \mu_{\Delta d} + \lambda \varepsilon_{\Delta c,t} - L_t \times 1 \{J_{\Delta c,t} \neq 0\} \quad (21)$$

$\lambda$  here represents leverage.  $1 \{\cdot\}$  is the indicator function. Dividends are thus modeled as permanently declining by an amount  $L_t$  on the occurrence of a disaster. The value of  $L$  is allowed to change over time and follows the process

$$L_t = (1 - \rho_L) \bar{L} + \rho_L L_{t-1} + \varepsilon_{L,t} \quad (22)$$

We calibrate  $\bar{L} = 0.5$  and  $\rho_L = 0.87^{1/12}$  as in the previous model, and  $\varepsilon_{L,t} \sim N(0, 0.04^2)$ , which means that the standard deviation of  $L$  is 0.25. We set the coefficient of relative risk aversion to 7 to match the Sharpe ratio on one-month variance swaps (as we did for the time-varying disaster model; note that for the long-run risk model, there was no need to adjust the calibration since the paper already targeted the behavior of the one-month variance swap). Other than the change in risk aversion, our calibration of the model is

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<sup>36</sup>The upper bound on risk aversion is a common feature of models in which the riskiness of the economy varies over time.

nearly identical to Gabaix’s (2012), which implies that we will retain the ability to explain the same ten puzzles that he examines.<sup>37</sup> He did not examine the ability of his model to match the term structure of variance claims, so this paper provides a new test of the theory.

## 5.2 Results

We now examine the implications of the three models for the variance forward curve. Figure 10 plots population moments from the models against the values observed empirically. The top panel reports annualized Sharpe ratios for forward variance claims with maturities from 1 to 12 months. Our calibration of Gabaix’s model with time-varying recovery rates matches the data well: it generates a Sharpe ratio for the one-month claim of -1.3, while all the forward claims earn Sharpe ratios of zero, similarly to what we observe in the data.

On the other hand, both models with Epstein–Zin preferences significantly underprice variance risk at the short end and, at the same time, overprice risk at the longer end of the variance curve. The Sharpe ratio on the one month forward is far smaller in these models (at about 0.3) than in the data (at about -1.3). By contrast, both models generate Sharpe ratios for claims on variance more than three months ahead that are counterfactually large, almost as large as the one-month forward. In the data, instead, they are zero or even positive at all horizons above 3 months.

The underpricing of risks at the short end is caused by the fact that these models do not generate pricing kernels sufficiently volatile to give *any* asset a Sharpe ratio of 1.3. However, simply increasing the volatility of the pricing kernel by increasing risk aversion will not solve the problem, as it will simply increase the Sharpe ratio at all maturities and exacerbate the mispricing at horizons longer than 3 months (more over, as we mentioned above, it is in fact not possible to raise risk aversion further in Wachter’s (2013) model).

The economic intuition for the result is straightforward. If investors are risk-averse, then periods of high expected future consumption volatility are periods of low lifetime utility. And under Epstein–Zin preferences, periods with low lifetime utility are periods with high marginal utility. Investors thus desire to hedge news about future consumption volatility, and in these models forward variance claims allow them to do so. Moreover, in these models volatility in all future periods (discounted at a rate close to the rate of pure time preference, and therefore close to 1 in standard calibrations) affects lifetime utility, which is why investors in these models pay nearly the same amount to hedge volatility at any horizon.

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<sup>37</sup>In particular, he shows that the model can generate a high equity premium, a low risk-free rate, excess volatility in stock returns, stock return predictability, a steep yield curve for nominal bonds, predictability in bond returns, large corporate credit spreads, a premium on out-of-the-money puts, predictability of stock returns from the put premium, and a value premium.

The expected returns on the variance claims are closely related to the average slope of the term structure. The bottom panel of Figure 10 reports the average term structure in the data and in the models. The figure shows, as we would expect, that neither model with Epstein–Zin preferences generates a curve that is as concave as we observe in the data. Instead, the DY model generates a curve that is too steep everywhere (including on the very long end), while the time-varying disaster model generates a curve that is too flat everywhere. On the other hand, the average term structure in the model with time-varying recovery rates qualitatively matches what we observe in the data – it is steep initially and then perfectly flat after the first month.

The comparison between the calibrated models and the data reported in Figure 10 does not take into account the statistical uncertainty due to the fact that we only observe variance swap prices in a specific sample. To directly test the models against the data, we simulate the calibrated models and verify how likely we would be to see a period in which the variance swap curve looks like it does in our data (similar to the analysis in van Binsbergen and Koijen (2015)). In particular, we focus on the ability of the models to match the steepness at both ends of the forward variance curve.

Table 8 reports results from those simulations. We examine 215-month simulations to compare to our full sample since 1996, and 70-month simulations to compare to the shorter sample in which we have 10-year swaps available. For each simulation, we calculate the averages of the simulated values of  $(F_t^3 - F_t^0)$ ,  $(F_t^{12} - F_t^3)$ , and, in the long sample,  $(F_t^{120} - F_t^3)$ . Table 8 reports the fraction of simulated samples in which the sample mean of  $(F_t^3 - F_t^0)$  is at least as *large* as we see in the data, the sample mean of  $(F_t^{12} - F_t^3)$  is *smaller* than in the data, or the sample mean of  $(F_t^{120} - F_t^3)$  is *smaller* than in the data. These fractions are one-sided p-values: they measure the probability that the model would have generated slopes as extreme as we observe in the data. Furthermore, the bottom rows report the fraction of samples in which the models simultaneously generate slopes as high as we observe below three months and as flat as we observe above three months. They are thus p-values for tests of whether the models can match the observed concavity of the term structure.

The long-run risk model does a relatively good job of generating a large slope at the short end – 20 percent of the long samples and 38 percent of the short samples are at least as steep as in our data. However, the slopes after the three-month maturity struggle to match the data – the sample mean of  $(F_t^{12} - F_t^3)$  is as small as observed empirically in the long sample less than 0.1 percent of the time. When we ask how many samples generate both the steep slope below three months and the flat slope after three months, the p-value is less than 0.005. In other words, the long-run risk model generates a large short-maturity slope,

but significantly fails to match the flatness of the term structure after three months.

The model with time-varying disaster risk and Epstein–Zin preferences has the opposite problem from the long-run risk model: it generates a relatively flat term structure at maturities longer than three months, but it fails to match the steep slope observed below three months. The p-values are similar to those for the long-run risk model – the probability that the time-varying disaster model generates the steep slope below three months is less than 0.1 percent, while the probability that it generates a slope as flat as we see beyond three months is 23 to 55 percent. In none of our simulations does the time-varying disaster model simultaneously match the slopes both below and above three months.

Finally, table 8 shows that the model with time-varying recovery can in fact match well both the slopes below and above three months. It has a slope as steep as we observe empirically between 0 and 3 months in 70 percent of the short samples and 82 percent of the long samples. It also has a slope after three months as flat as we observe empirically in 100 percent of the samples. It therefore matches the slopes both below and above three months in 69 and 80 percent of the short and long samples, respectively.

To summarize, then, we can reject the long-run risk and time-varying disaster models with p-values of less than 1 percent, while the time-varying recovery model is not rejected. We thus take the results in figure 10 and table 8 as providing further support for Gabaix’s model of time-varying recovery rates.

The main features of the models that affect their ability to match our data can be summarized as follows. In models with Epstein–Zin preferences where agents have preferences for early resolution of uncertainty, investors will pay to hedge shocks to expected future consumption volatility, especially at long horizons. If the equity market is modeled as being related to a consumption claim, then long-term forward variance claims should have large negative returns because they provide hedge volatility news. But in the data, we observe shocks to future expected volatility and find that their price is close to zero.

While it is true that there exist parameterizations of Epstein–Zin preferences for which agents are *not averse* to bad news about future expected volatility, or even *enjoy* news about high future volatility, these are degenerate or nonstandard cases (in the former, the model collapses to power utility, and in the latter, agents have preference for late resolution of uncertainty). The very motivation behind using Epstein–Zin preferences in asset pricing models is to model investors who are averse to bad news about the future, i.e. agents that have an intertemporal hedging motive. It is that force, generated by standard calibration of Epstein–Zin preferences, with preference for early resolution of uncertainty, that is at odds with the term structure of variance swaps.

Models with power utility, or where the variation in expected stock market volatility is independent of consumption volatility, solve that problem since investors are myopic and shocks to future expected volatility are not priced. However, the models also need to explain the high risk price associated with the realized volatility shock. In a power utility framework, this can be achieved if states of the world with high volatility are associated with large drops in consumption, as in a disaster model.

### 5.3 The behavior of volatility during disasters

In order for variance swaps to be useful hedges in disasters, realized volatility must be high during large market declines. A number of large institutional asset managers sell products meant to protect against tail risk that use variance swaps, which suggests that they or their investors believe that realized volatility will be high in future market declines.<sup>38</sup>

In the spirit of Barro (2006), we now explore the behavior of realized volatility during consumption disasters and financial crises using a panel data of 17 countries, covering 28 events. We obtain two results. First, volatility is indeed significantly higher during disasters. Second, the increase in volatility is not uniform during the disaster period; rather, volatility spikes for one month only during the disaster and quickly reverts. It is those short-lived but extreme spikes in volatility that make variance swaps a good product to hedge tail risk.

We collect daily market return data from Datastream for a total of 37 countries since 1973. We compute realized volatility in each month for each country. To identify disasters, we use both the years marked by Barro (2006) as consumption disasters and the years marked by Schularick and Taylor (2012), Reinhart and Rogoff (2009) and Bordo et al. (2001) as financial crises.<sup>39</sup> Given the short history of realized volatility available, our final sample contains 17 countries for which we observe realized volatility and that experienced a disaster during the available sample. Table 9 shows for each country the first year of our *RV* sample and the years we identify as consumption or financial disasters.

The first three columns of Table 9 compare the monthly annualized realized volatility during disaster and non-disaster years. Column 1 shows the maximum volatility observed in any month of the year identified as a disaster averaged across all disasters for each country. Column 2 shows the average volatility during the disaster years, and column 3 shows the average volatility in all other years.

Comparing columns 2 and 3, we can see that in almost all cases realized volatility is indeed

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<sup>38</sup>In particular, see Man Group's TailProtect product (Man Group (2014)), Deutsche Bank's ELVIS product (Deutsche Bank, 2010) and the JP Morgan Macro Hedge index.

<sup>39</sup>See Giglio et al. (2014) for a more detailed description of the data sources.

higher during disasters. For example, in the US the average annualized realized volatility is 25 percent during disasters and 15 percent otherwise. Column 1 reports the average across crises of the highest observed volatility. Within disaster years there is large variation in realized volatility: the maximum volatility is always much higher than the average volatility, even during a disaster. Disasters are associated with large spikes in realized volatility, rather than a generalized increase in volatility during the whole period.

To confirm this result, in Figure 11 we perform an event study around the peak of volatility during a disaster. For each country and for each disaster episode, we identify the month of the volatility peak during that crisis (month 0) and the three months preceding and following it. We then scale the volatility behavior by the value reached at the peak, so that the series for all events are normalized to 1 at the time of the event. We then average the rescaled series across our 28 events.

The figure shows that indeed, the movements in volatility that we observe during disasters are short-lived spikes, where volatility is high for essentially only a single month. In the single months immediately before and after the one with the highest volatility, volatility is 40 percent lower than its peak, and it is lower by half both three months before and after the worst month.

## 6 Conclusion

This paper shows that it is only the transitory part of realized variance that is priced. That fact is not consistent with a broad range of structural asset pricing models. It is qualitatively consistent with a model in which investors desire to hedge rare disasters, but not news about the future probability of disaster. Interestingly, the data is not consistent with all disaster models. The key feature that we argue models need in order to match our results is that variation in expected stock market volatility is not priced by investors, whereas the transitory component of volatility is strongly priced.

The idea that variance claims are used to hedge crashes is consistent with the fact that many large asset managers, such as Deutsche Bank, JP Morgan, and Man Group sell products meant to hedge against crashes that use variance swaps and VIX futures. These assets have the benefit of giving tail protection, essentially the form of a long put, but also being delta neutral (in an option-pricing sense). They thus require little dynamic hedging and yield powerful protection against large declines.

More broadly, shocks to expected volatility, such as that observed during the recent debt ceiling debate, are a major driving force in many current macroeconomic models. If

aggregate volatility shocks are a major driver of the economy, we would expect investors to desire to hedge them. We find, though, that the average investor is indifferent to such shocks. The evidence from financial markets is thus difficult to reconcile with the view that volatility shocks are an important driver of business cycles or welfare.

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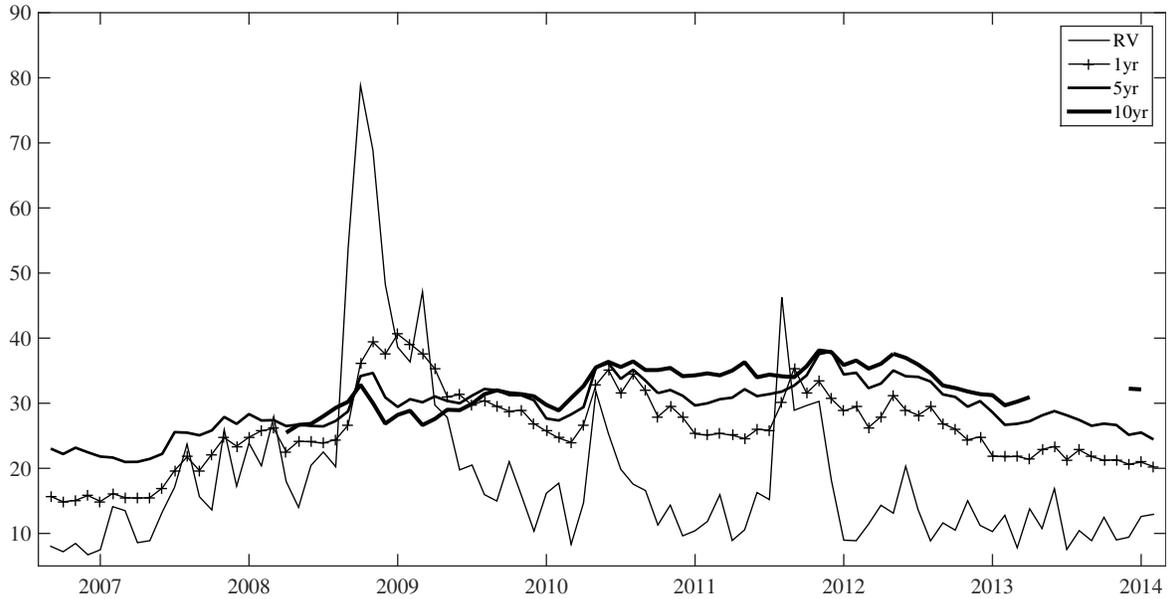
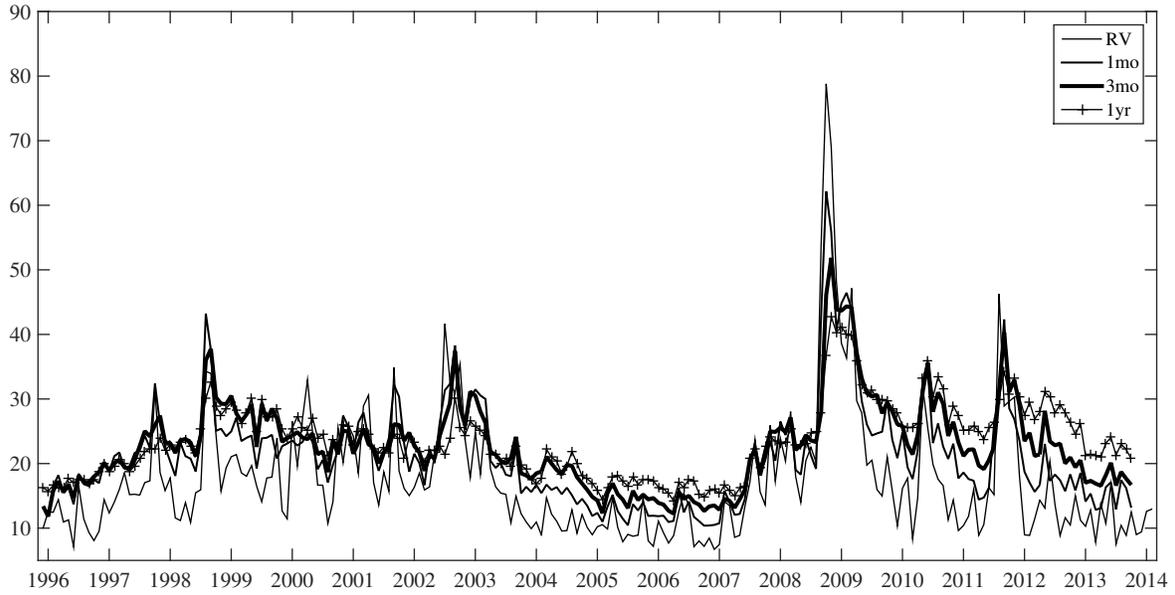
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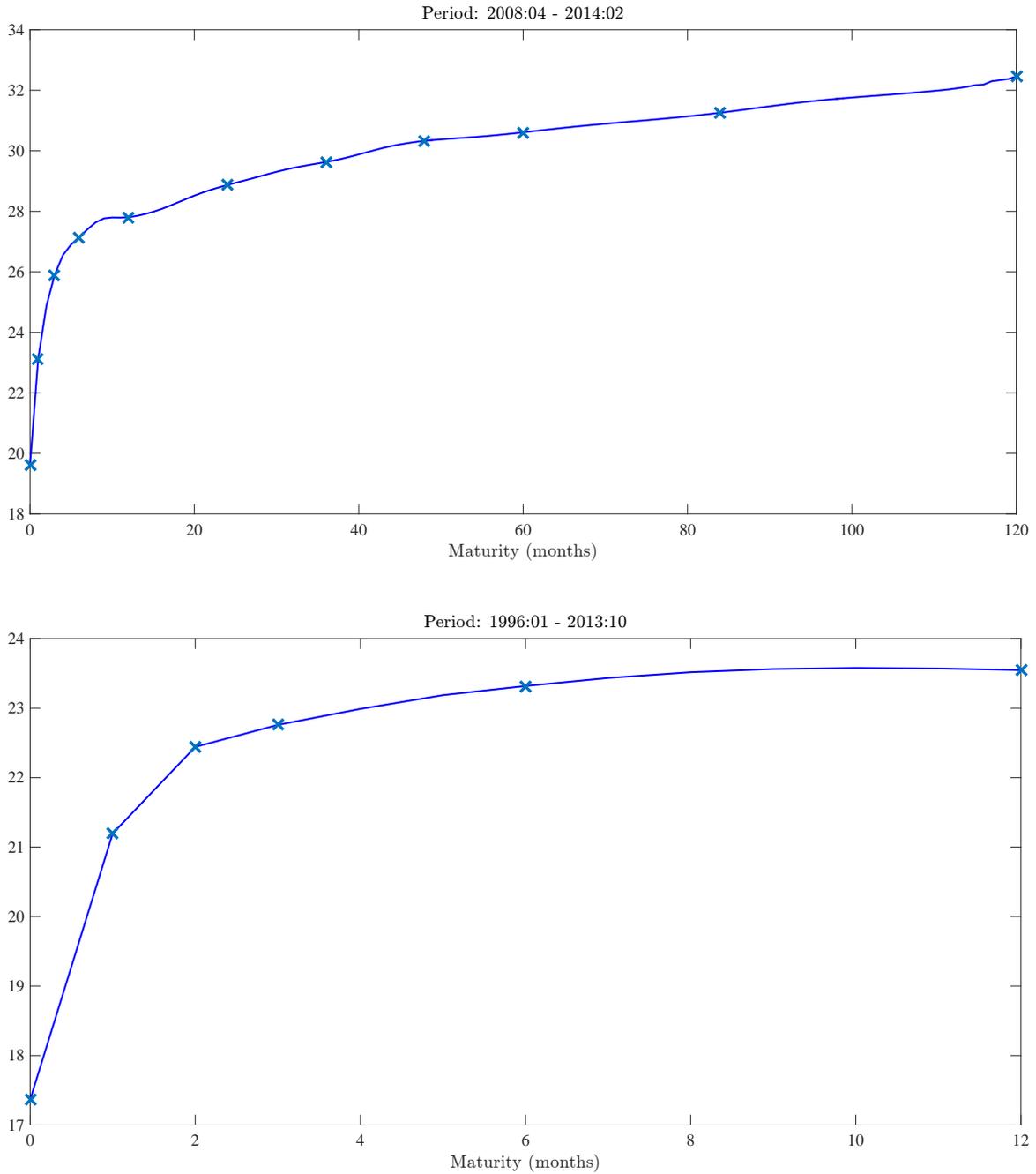
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Figure 1: Time series of forward variance claim prices



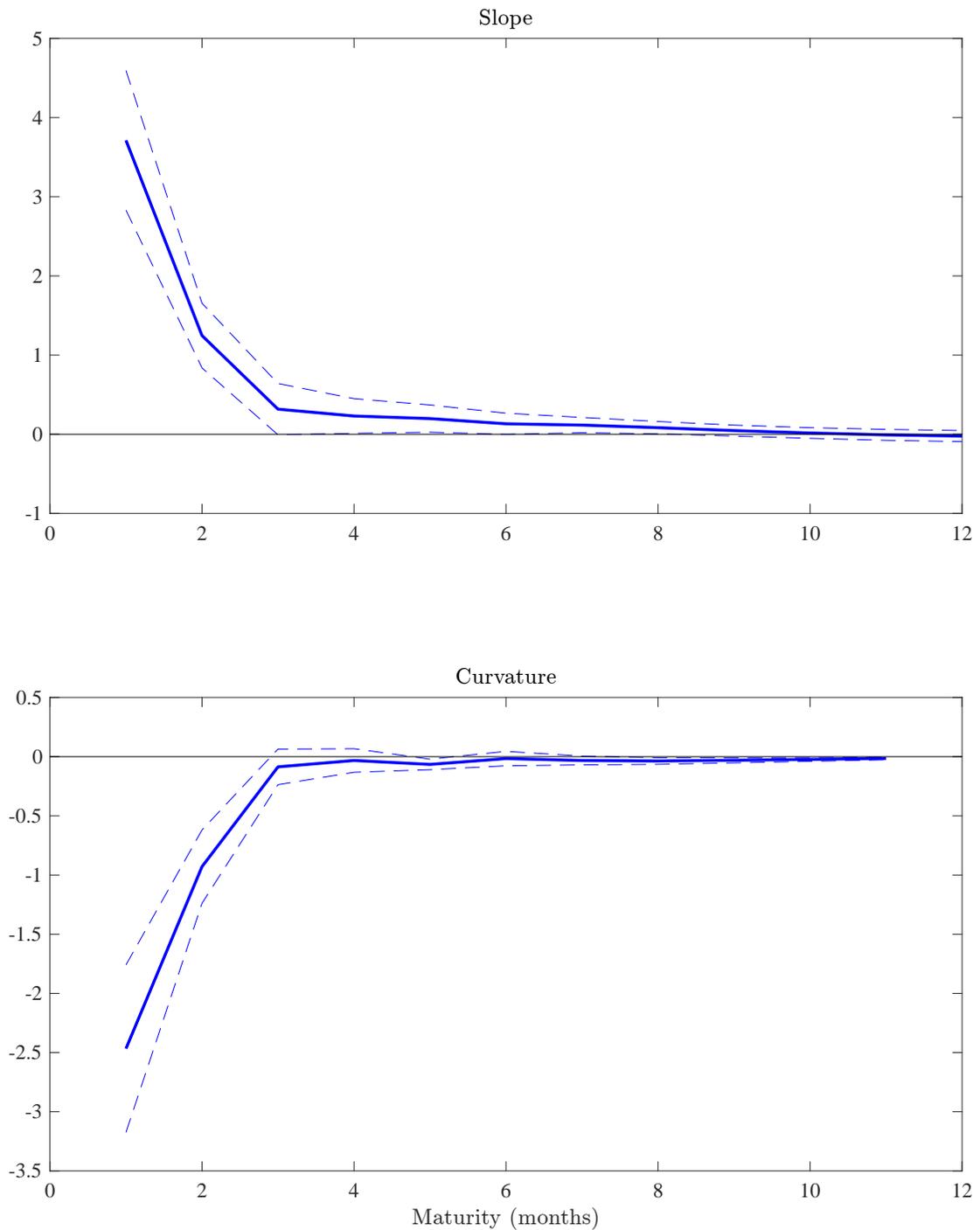
**Note:** The figure shows the time series of forward variance claim prices of different maturities. For readability, each line plots the prices in annualized volatility terms,  $100 \times \sqrt{12 \times F_t^n}$ , for a different  $n$ . The top panel plots forward variance claim prices for maturities of 1 month, 3 months, and one year. The bottom panel plots forward variance claim prices for maturities of 1 year, 5 years and 10 years. Both panels also plot annualized realized volatility,  $100 \times \sqrt{12 \times F_t^0}$ .

Figure 2: Average forward variance claim prices



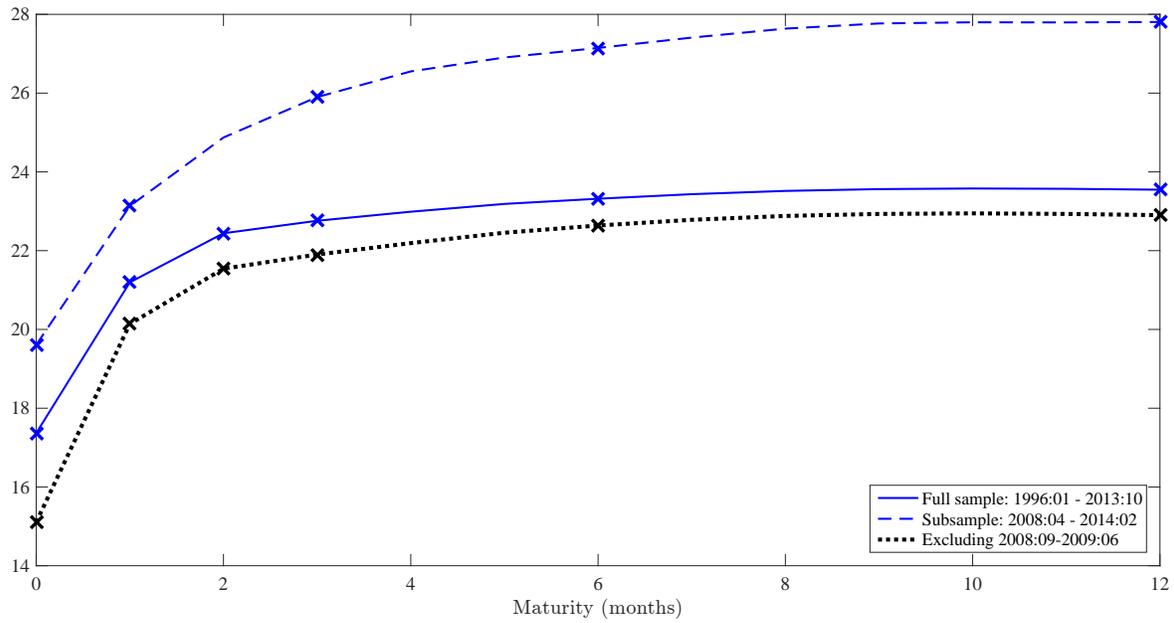
**Note:** The figure shows the average prices of forward variance claims of different maturity, across different periods. The top panel shows average prices between 2008 and 2013, when we observe maturities up to 10 years (Dataset 2). The bottom panel shows averages between 1996 and 2013, for claims of up to 1 year maturity (Dataset 1). In each panel, the "x" mark prices of maturities we directly observe in the data (for which no interpolation is necessary). All prices are reported in annualized volatility terms,  $100 \times \sqrt{12 \times F_t^n}$ . Maturity zero corresponds to average realized volatility,  $100 \times \sqrt{12 \times F_t^0}$ .

Figure 3: Slope and curvature of the term structure of forward variance claims



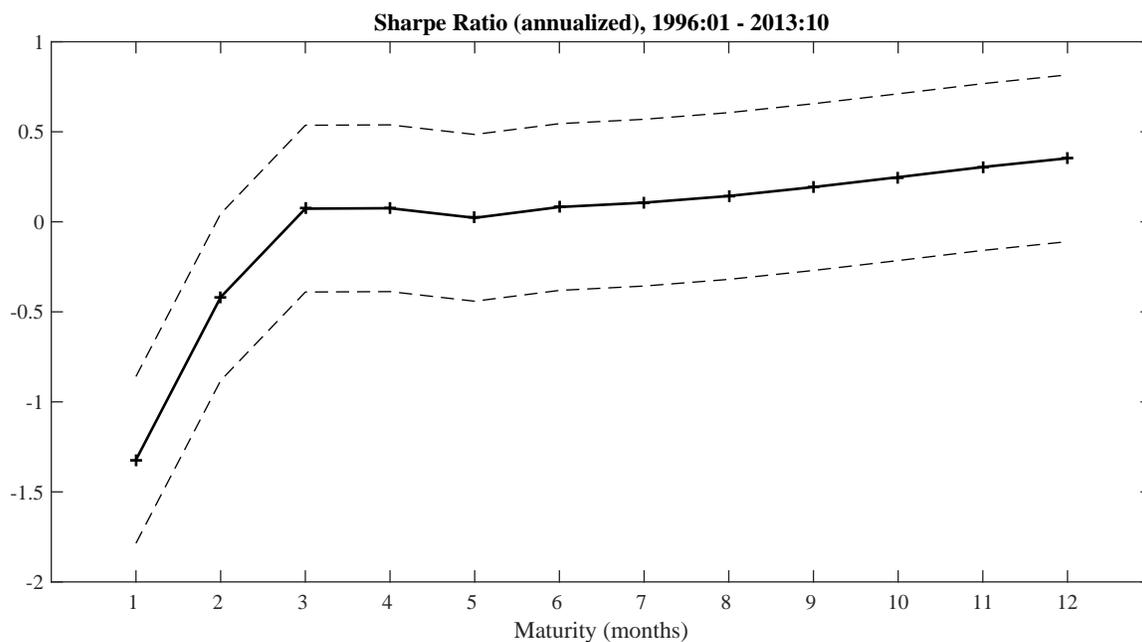
**Note:** The top panel plots the slope of the term structure of variance swaps (figure 2) at each maturity. The bottom panel plots the curvature of the same curve at each maturity. Dotted lines are 95% confidence intervals constructed using Newey-West with 6 lags.

Figure 4: Subsample analysis of forward variance claims



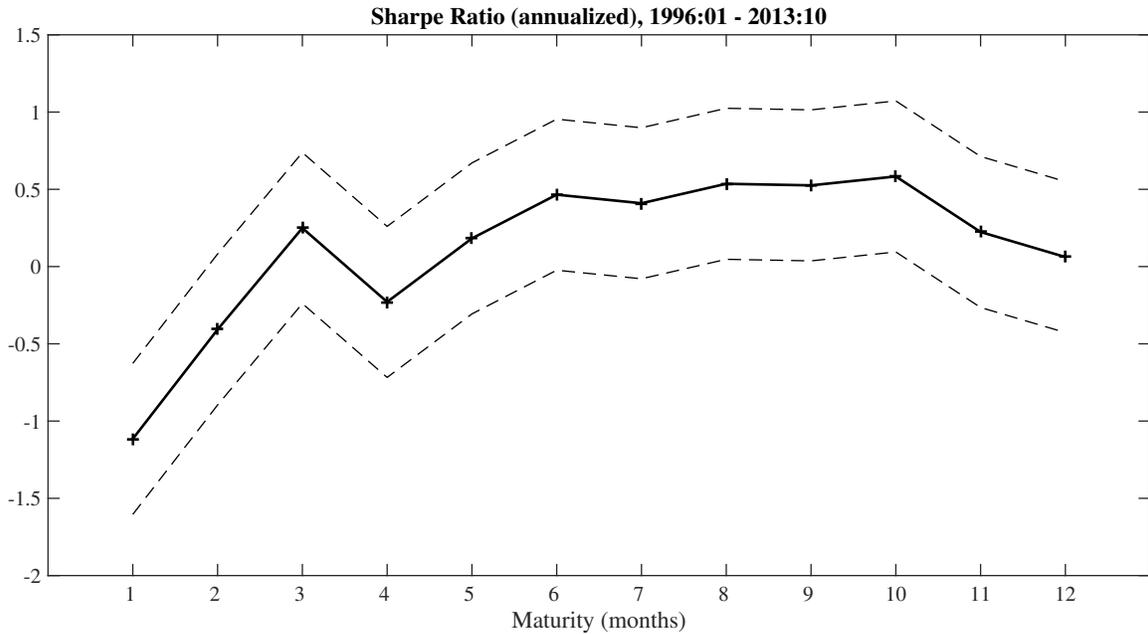
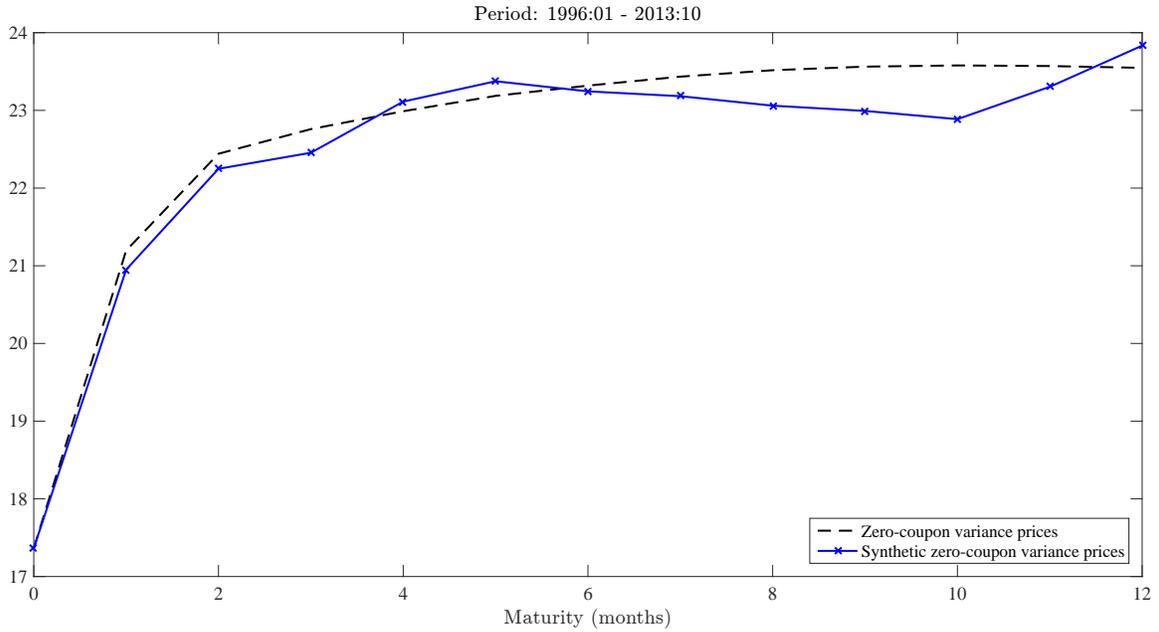
**Note:** The figure compares the average prices of forward variance claims for maturities up to 1 year, for the two subsamples of the top and bottom panel of figure 2, as well as for the period that excludes the financial crisis.

Figure 5: Annualized Sharpe ratios for forward variance claims



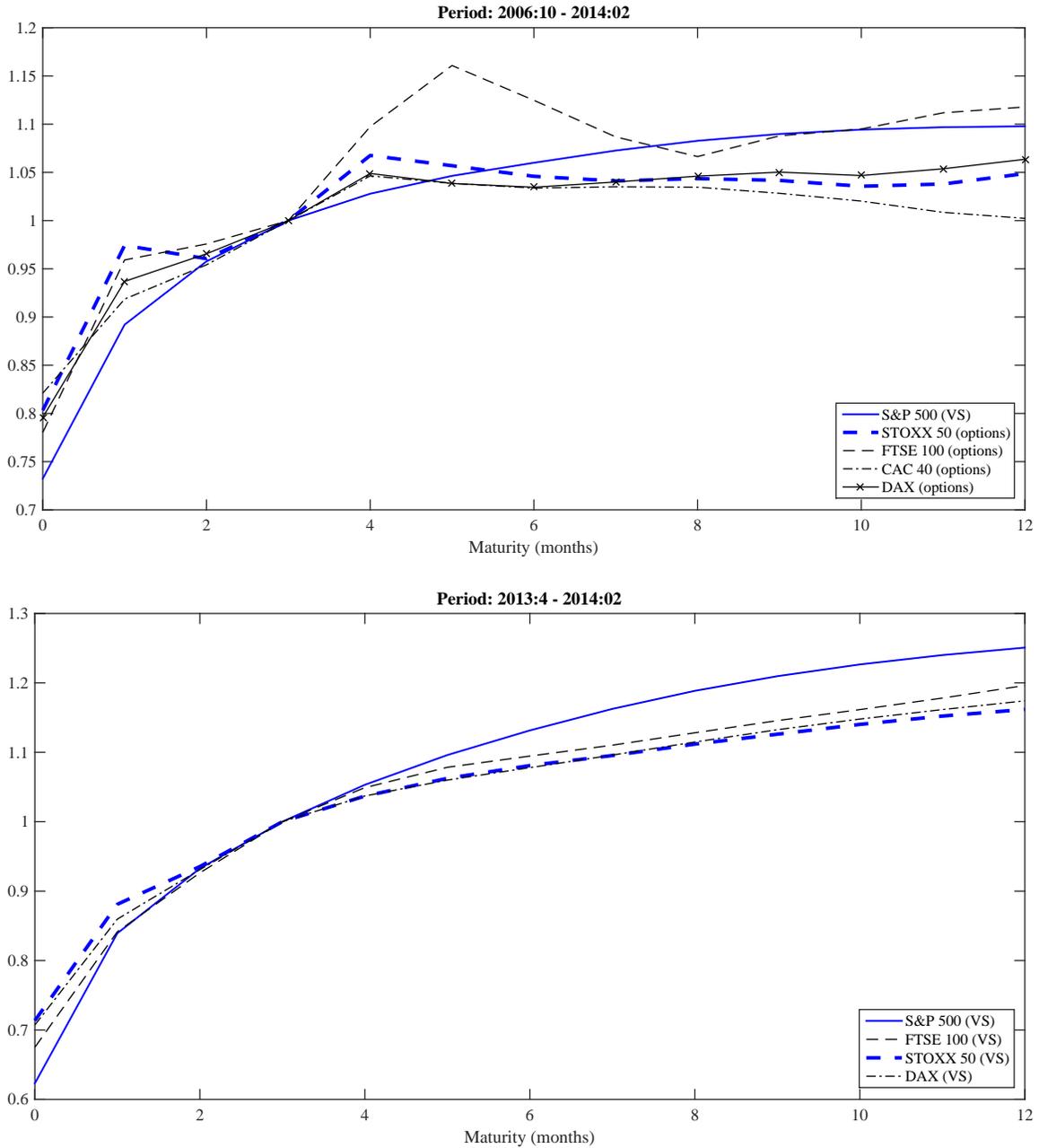
**Note:** The figure shows the annualized Sharpe ratio for the forward variance claims. The returns are calculated assuming that the investment in an n-month variance claim is rolled over each month. Dotted lines represent 95% confidence intervals. All tests for the difference in Sharpe ratio between the 1-month variance swap and any other maturity confirm that they are statistically different with a p-value of 0.03 (for the second month) and  $< 0.01$  (for all other maturities). The sample used is 1996-2013.

Figure 6: Synthetic forward variance claims: prices and Sharpe ratios



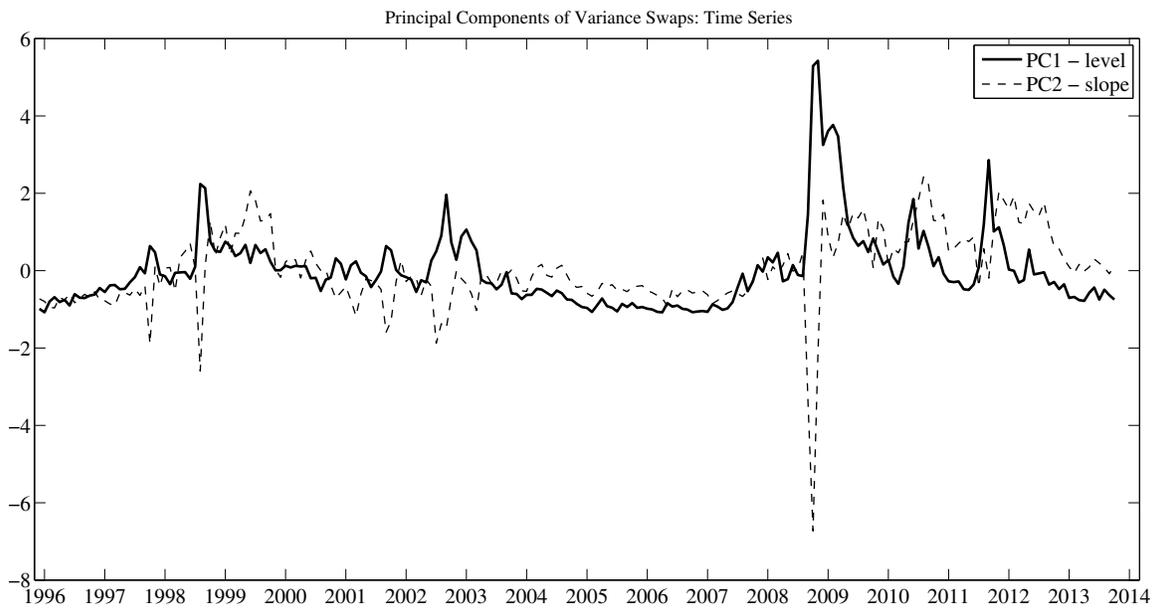
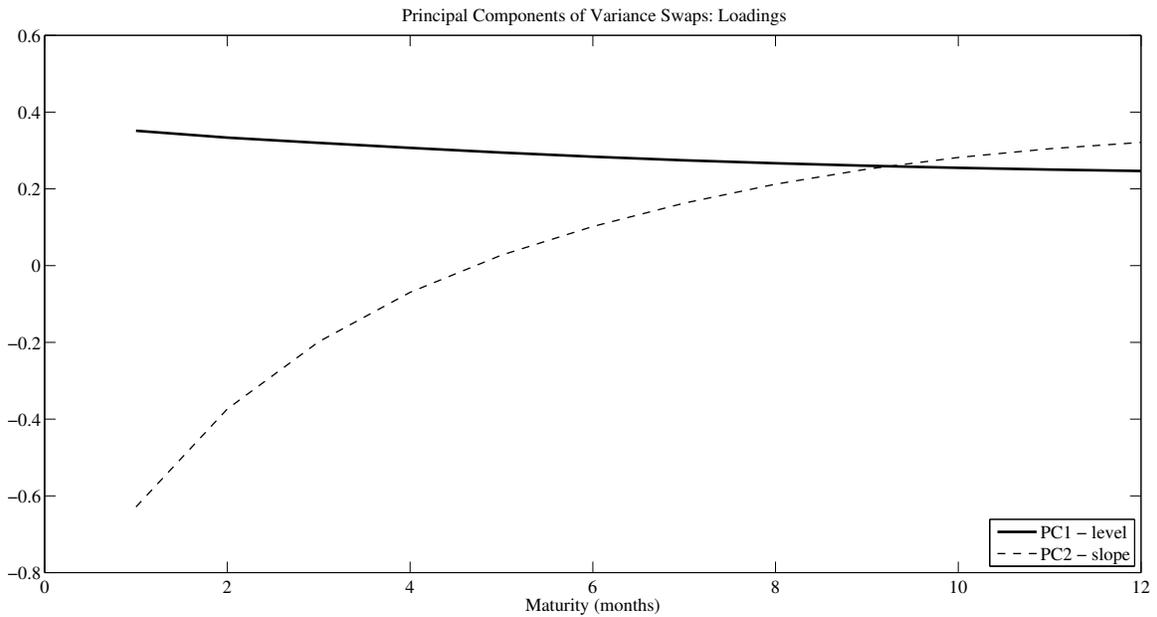
**Note:** See Figure 2. The solid line in the top panel plots average prices of forward variance claims calculated using the formula for the VIX index and data on option prices from the CBOE. The dotted line is the set of average prices of forward variance claims constructed from variance swap prices. The bottom panel plots annualized Sharpe ratios for forward variance claims returns with prices calculated using the VIX formula and CBOE option data. Dotted lines in the bottom panel represent 95% confidence intervals. The sample covers the period 1996-2013.

Figure 7: Average forward variance claim prices for international markets



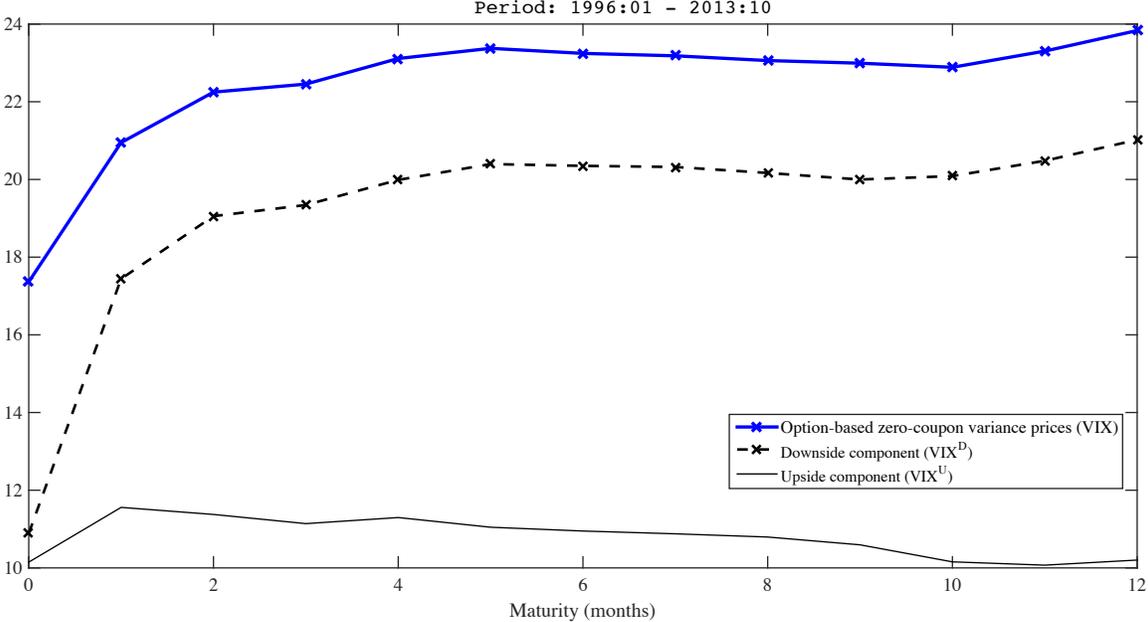
**Note:** The figure plots the average prices of forward variance claims as in Figure 2 for different international indices. The series for the S&P 500 (both in the top and bottom panel) is obtained from variance swaps (as in Figure 2). The top panel shows international curves obtained using option prices, using the same methodology used to construct the VIX for the S&P 500 (as in Figure 6). Options data is from OptionMetrics. The series cover FTSE 100, CAC 40, DAX, and STOXX 50, for the period 2006-2014. The bottom panel shows international curves obtained using variance swaps on the FTSE 100, DAX, and STOXX 50, for one year starting in April 2013. All series are rescaled relative to the price of the 3-month forward variance price.

Figure 8: Principal components of variance swap prices



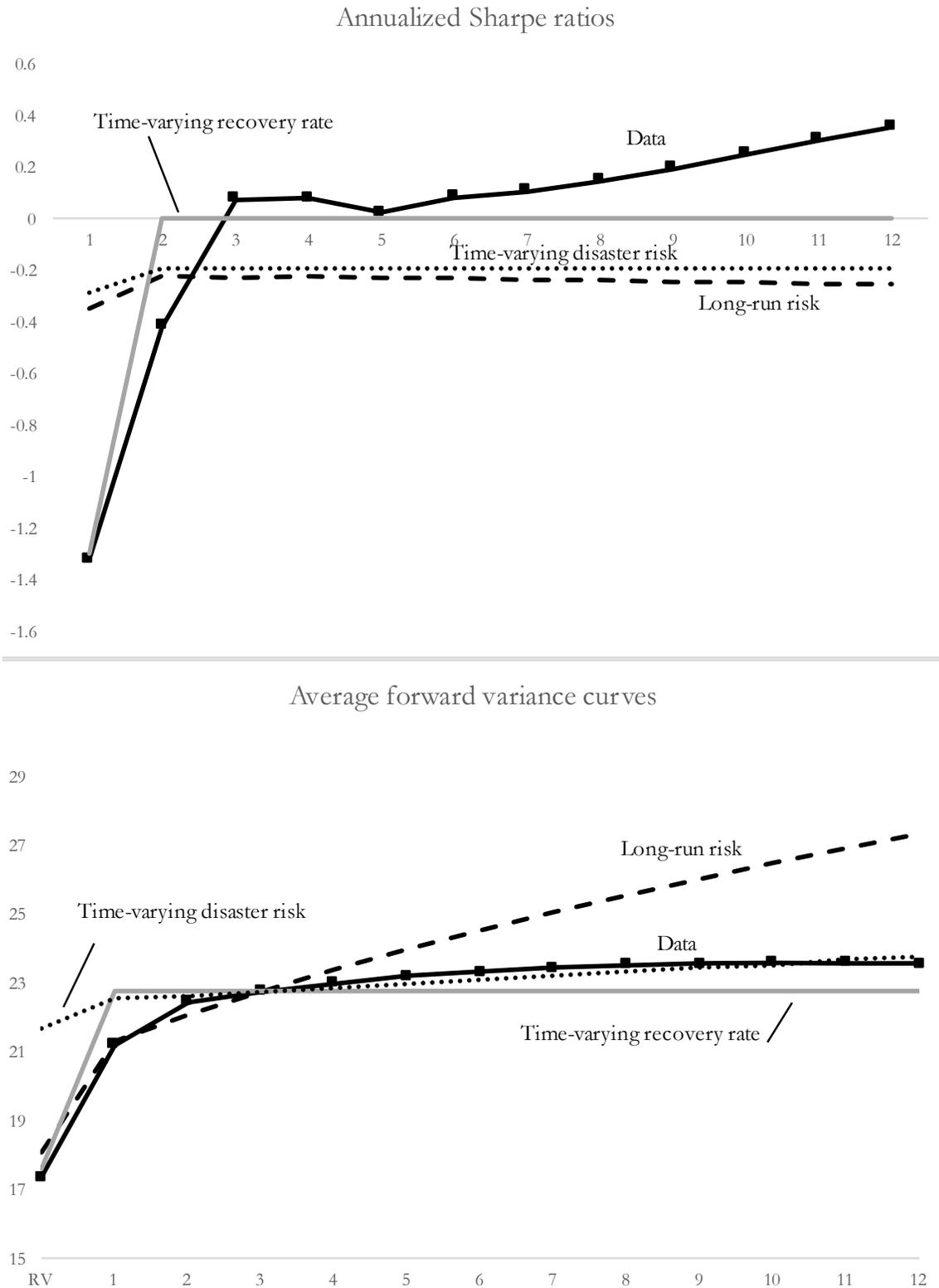
**Note:** The top panel plots the loadings of the variance swap prices on the level and slope factors (first two principal components). The bottom panel plots the time series of the level and slope factors. Both are normalized to have zero mean and unit standard deviation and are uncorrelated in the sample. The sample covers the period 1996-2013.

Figure 9: Decomposing the upward and downward volatility components



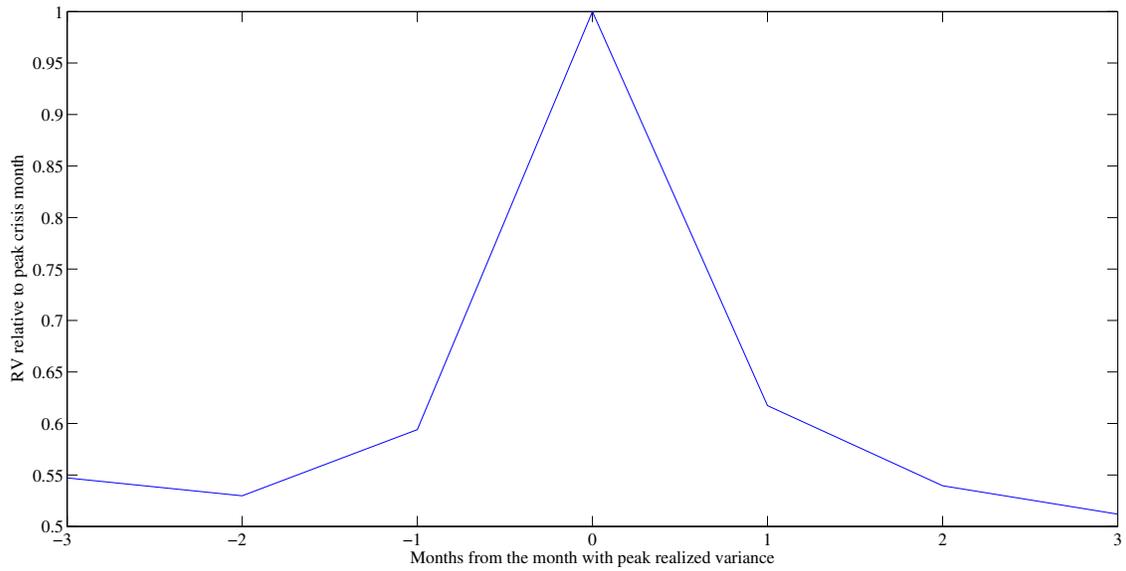
**Note:** The solid thick line plots average prices of forward variance claims calculated using the formula for the VIX index. The dashed line plots the forward prices of the downside component of the VIX,  $VIX^D$ . The thin solid line plots the forward prices of the upside component of the VIX,  $VIX^U$ . All series are constructed using option data from CBOE. The sample covers the period 1996-2013.

Figure 10: Sharpe ratios and average term structure in different models



**Note:** The top panel gives the population Sharpe ratios from the three models and the sample values from the data. The bottom panel plots population means of the prices of forward claims. All the curves are normalized to have the same value for the 3-month forward claim.

Figure 11: Average behavior of RV during consumption disasters and financial crises



**Note:** We calculate realized variance in each month of a crisis and scale it by the maximum realized variance in each crisis. The figure plots the average of that scaled series for each country and crisis in terms of months relative to the one with the highest realized variance.

Table 1: Volume of variance swaps across maturities

Maturity (months)	Volume (million vega)	Volume (percentage)
1	402	6%
2	403	6%
3	78	1%
4-6	1037	14%
7-12	1591	22%
13-24	2371	33%
25-60	1315	18%
60+	48	1%
Total	7245	100%

**Note:** Total volume of variance swap transactions occurred between March 2013 and June 2014 and collected by the DTCC.

Table 2: Characteristics of returns

Maturity (months)	Mean	Std	Min	25th p.	Median	75th p.	Max	Skew	Exc.Kurt.
1	-25.7	67.9	-85.5	-58.4	-40.2	-16.0	686.4	6.2	56.5
2	-5.8	47.7	-59.3	-32.9	-18.4	9.3	376.0	3.9	23.4
3	0.7	33.9	-46.1	-21.4	-5.3	14.5	249.4	2.7	14.1
4	0.6	27.4	-42.2	-17.3	-5.6	11.2	170.4	2.0	7.6
5	0.1	22.5	-37.3	-14.0	-3.7	9.8	126.7	1.6	5.2
6	0.5	19.6	-31.0	-12.2	-3.8	12.9	100.6	1.3	3.3
7	0.6	18.6	-31.4	-12.4	-2.5	11.0	90.7	1.1	2.5
8	0.7	17.4	-29.8	-11.4	-2.9	11.6	81.6	1.0	2.0
9	0.9	16.2	-27.7	-10.2	-1.9	9.2	74.6	0.9	1.7
10	1.1	15.6	-30.0	-9.6	-2.0	9.8	70.8	0.9	1.5
11	1.4	16.0	-32.6	-9.9	-1.9	11.2	69.7	0.9	1.3
12	1.8	17.4	-35.0	-10.3	-2.4	12.1	70.4	1.0	1.4

**Note:** The table reports descriptive statistics of the monthly returns for forward variance claims (in percentage points). For each maturity  $n$ , returns are computed each month as  $R_{t+1}^n = \frac{F_{t+1}^{n-1} - F_t^n}{F_t^n}$ . Given the definition that  $F_t^0 = RV_t$ , the return on a one-month claim,  $R_{t+1}^1$  is the percentage return on a one-month variance swap.

Table 3: Reduced-form pricing estimates

<i>Panel A: Betas</i>				
		Shock 1	Shock 2	Pure RV
	1	0.37	-0.19	0.33
	2	0.39	-0.12	0.05
	3	0.28	0.00	-0.02
	4	0.22	0.04	-0.02
	5	0.18	0.05	-0.01
Maturities (months)	6	0.15	0.06	0.00
	7	0.13	0.06	0.00
	8	0.12	0.06	0.00
	9	0.11	0.06	0.00
	10	0.11	0.06	0.00
	11	0.11	0.06	0.00
	12	0.11	0.06	0.00
<i>Panel B: Risk Prices</i>				
	Risk prices	-0.67	-0.47	-2.69***
	Standard error	0.45	0.66	0.46
	Difference from Pure RV (p-value)	< .001	< .001	
	$R^2$		0.999	

**Note:** Results of Fama–MacBeth regressions using the 12 forward claims as test assets and the three rotated VAR innovations as pricing factors. Shock 1 has effects on all three factors; shock 2 affects only the slope and RV, and pure RV only affects RV on impact. The top section reports betas on the three factors. The bottom section reports estimated risk prices and the Fama–MacBeth standard errors. \*\*\* denotes significance at the 1-percent level. Risk prices are annualized by multiplying by  $\sqrt{12}$ .

Table 4: Pricing results for the CAPM

	(1)	(2)		
	CAPM	CAPM + pure RV		
<b>Panel A: Betas</b>				
	Rm	Pure RV	Rm	
1	-7.21	0.35	-7.74	
2	-6.93	0.07	-7.03	
3	-5.04	0.00	-5.04	
4	-3.97	0.00	-3.96	
5	-3.14	0.00	-3.15	
VS, maturities	6	-2.59	0.01	-2.60
	7	-2.32	0.01	-2.34
	8	-2.13	0.01	-2.15
	9	-2.03	0.01	-2.04
	10	-2.01	0.01	-2.02
	11	-2.06	0.01	-2.07
	12	-2.17	0.00	-2.17
Market	1.00	0	1.00	
<b>Panel B: Risk Prices</b>				
Risk prices	0.0106***	-0.69***	0.0051	
Standard error	0.0034	0.11	0.0033	
$R^2$	0.383	0.998		

**Note:** See Table 3. Pricing results for the CAPM and for the CAPM with pure RV. The test assets are the 12 forward variance claims and the market portfolio. The market portfolio is given 12 times as much weight as the variance claims to ensure that it is priced correctly in the estimation.

Table 5: Forecasting volatility at different horizons: R2

Predictor: with $PE_t, DEF_t$	monthly $RV_{t+n}$		yearly $RV_{t+n}$		yearly $\Delta d_{t+n}$			
	$RV_t$	$\sqrt{RV_t}$	$RV_t$	$\sqrt{RV_t}$	$\Delta d_t$	$\sqrt{\Delta d_t}$		
Months			Years			Years		
1	0.39	0.45	1	0.41	0.56	1	0.00	0.09
2	0.21	0.34	2	0.10	0.25	2	0.00	0.02
3	0.18	0.32	3	0.05	0.09	3	0.06	0.07
6	0.15	0.26	5	0.02	0.04	5	0.05	0.07
12	0.10	0.18	10	0.00	0.21	10	0.02	0.03

**Note:** The first column of the table reports R2 of predictive regressions of monthly volatility  $n$  months ahead at the monthly frequency. The second column reports R2 of predictive regressions of yearly volatility  $n$  years ahead at the yearly frequency. The second column reports R2 of predictive regressions of yearly log dividend growth  $n$  years ahead at the yearly frequency. The left side of each column reports univariate regressions using the lagged value of the target, while the right side of each column adds the market price-earnings ratio and the default spread as predictors. The sample is 1926-2014.

Table 6: Average prices and pricing errors for the no-arbitrage model

Maturity (months)	Sample		Fitted		RMSE
	Mean	Std	Mean	Std	
1	21.24	8.00	21.70	7.86	1.00
2	21.89	7.55	21.88	7.55	0.33
3	22.22	7.25	22.04	7.30	0.40
6	22.75	6.64	22.42	6.74	0.63
12	23.20	6.06	22.99	6.11	0.53
24	23.65	5.58	23.87	5.45	0.61
>24					0.87

**Note:** Prices are reported in annualized volatility terms. The RMSE is calculated using the deviation of the fitted price from the sample price in annualized volatility terms.

Table 7: Steady-state risk prices

Sources of risk	Estimates	Standard Error
$s_t^2$ -risk	-0.11	0.31
$l_t^2$ -risk	-0.18	0.12
RV-risk	-1.70***	0.48

**Note:** Estimates of steady-state risk prices from the no-arbitrage model. Risk prices are annualized by multiplying by  $\sqrt{12}$ . \*\*\* denotes significance at the 1-percent level.

Table 8: Model tests using the variance swap data

	70-month simulations, up to 12mo maturity		
	Long-run risks	Time-varying disasters	Time-varying recovery
	p-value	p-value	p-value
Simulated 3mo/RV slope $\geq$ empirical slope	0.20	<0.01	0.69
Simulated slope 12mo/3mo $\leq$ empirical slope	<0.01	0.23	1.00
Simulated slope 120mo/3mo $\leq$ empirical slope	-	-	-
Joint test: 3mo/RV $\geq$ data and 12mo/3mo $\leq$ data	<0.01	<0.01	0.69
Joint test: 3mo/RV $\geq$ data and 120mo/3mo $\leq$ data	-	-	-
	215-month simulations, up to 120mo maturity		
	Long-run risks	Time-varying disasters	Time-varying recovery
	p-value	p-value	p-value
Model 3mo/RV slope $\geq$ empirical slope	0.38	<0.01	0.82
Model slope 12mo/3mo $\leq$ empirical slope	0.05	0.55	1.00
Model slope 120mo/3mo $\leq$ empirical slope	0.02	0.12	1.00
Joint test: 3mo/RV $\geq$ data and 12mo/3mo $\leq$ data	<0.01	<0.01	0.82
Joint test: 3mo/RV $\geq$ data and 120mo/3mo $\leq$ data	<0.01	<0.01	0.82

**Note:** Notes: We simulate 10,000 70- and 215-month samples from the three models (respectively, in the top and bottom panels). In each simulation, we calculate 3-0 (RV), 12-3, and 120-3 month slopes of the variance forward term structure. The numbers in the first row of each panel are the fraction of samples in which the models generate a slope at the short end of the curve at least as large as observed empirically. The second and third rows are the fraction of samples in which the models generate slopes at the long end of the curve at least as flat as observed empirically. The bottom rows are the fraction of samples in which both conditions are satisfied.

Table 9: Realized volatility during disasters

Country	Peak Vol. during disaster	Mean Vol. during disaster	Mean Vol. outside disaster	Sample start year	Consumption disasters	Financial crises
US	47.5	25.2	14.9	1926	1933	1929, 1984, 2007
UK	24.6	16.4	15.1	1973		1974, 1984, 1991, 2007
France	72.1	31.4	16.6	1973		2008
Japan	40.9	21.4	15.1	1973		1992
Australia	33.7	13.8	15.1	1973		1989
Germany	83.1	28.1	14.3	1973		2008
Italy	55.1	23.0	19.2	1973		1990, 2008
Sweden	52.3	27.7	19.5	1982		1991, 2008
Switzerland	67.1	27.4	12.1	1973		2008
Belgium	66.1	32.0	12.4	1973		2008
Finland	29.3	18.9	25.0	1988	1993	1991
South Korea	80.0	43.6	24.6	1987	1998	1997
Netherlands	77.7	33.2	14.7	1973		2008
Spain	69.4	30.5	17.1	1987		2008
Denmark	37.2	14.7	14.4	1973		1987
Norway	44.2	20.2	20.7	1980		1988
South Africa	36.9	17.8	18.5	1973		1977, 1989

**Note:** Characteristics of annualized monthly realized volatility during and outside disasters across countries. Returns data used to construct realized volatility for the US is from CRSP, for all other countries from Datastream. Consumption disaster dates are from Barro (2006). Financial crisis dates are from Schularick and Taylor (2012), Reinhart and Rogoff (2009) and Bordo et al. (2001).

## A.1 Data quality

In this paper, we introduce two new datasets on variance swaps. Given that the datasets are new in the literature, we perform here a number of tests to ensure the quality of the data.

Dataset 2 (which contains monthly quotes from Markit, aggregated from many dealers) is constructed in the same way and by the same company as the CDS dataset from the same firm; this CDS data is known to be a high-quality dataset, and is the most widely used dataset in research on credit default swaps (Mayordomo, Pena, and Schwartz (2014)). In addition, Markit provides data on the number of quotes obtained from individual dealers, a measure of the depth of the market. The average number of quotes (11) is in the same range (8-15) of the typical number of quotes for CDS spreads, indicating that approximately as many large dealers trade in variance swaps as they trade in CDS.

Next, we note that unlike in the case of CDS, for the case of variance claims we actually observe the prices in many related markets, which we use to validate our variance swap data.

Our data quality check follows four main steps.

1. We confirm that in both data sets prices display a large amount of month to month variation, and the autocorrelations of price changes are close to zero for all maturities (non-zero autocorrelations would indicate that prices are stale).
2. We check that the data in Dataset 1 and Dataset 2 correspond almost perfectly for the dates and the maturities for which they overlap.
3. We check that our quotes correspond to actual trades we can observe for a period of time (using data from DTCC on actual transactions).
4. We check the correspondance between variance swap, VIX, and VIX futures data, which overlap for a substantial amount of time and maturities.

### A.1.1 Check 1: Price changes in the two datasets

A first data quality check is to ensure that the quotes and prices are updated in a timely way and do not contain stale information.<sup>1</sup> To begin with, we verify that prices always change month to month, across all datasets, all securities, and all months (with one single exception in one month and for one maturity only out of 16,928 observations, i.e., in 0.006%

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<sup>1</sup>For this analysis, we focus only on quotes we actually observe from the source, with no interpolation. Interpolation is however needed for Markit data after 2008, where we observe fixed calendar date maturities. For the Markit dataset, we therefore focus on the maturities around which we see most quotes, so that the interpolation needed is minimal: 1,2,3, 6, 12, 24, 36, 60, 84, 120.

of the data). This reflects the fact that our quotes are updated both by the dealers providing quotes to Markit (Dataset 2), and by the hedge fund that kept the price records forming Dataset 1.

A second check is whether prices changes are autocorrelated in any apparent way, in any dataset and for any maturity (we have 6 maturities available in the raw data for the first dataset, and 10 for the second one). Among the 16 cases, none of the autocorrelations are greater than 0.14, nor statistically significant even at the 10% level. Delayed or incomplete adjustment of reported prices may result in significant measured autocorrelation of price changes, but we find no evidence of this in our data.

### **A.1.2 Check 2: Dataset 1 vs. Dataset 2**

Next, we check the two datasets against each other. Dataset 1 contains monthly data since 1996, whereas Dataset 2 contains monthly data since 2006. For the period 2006-2013, the two datasets overlap.

The average correlation of prices in the two data sets is 0.999, with the minimum correlation for any maturity occurring for the 30-day maturity, where it drops to 0.997. Price changes are also extremely highly correlated, mostly above 0.99 (with the minimum being 0.98).

We also check whether either of the two data sets predicts price changes in the other, a sign of differing quality among the data sets. No price change in one data set significantly or economically predicts price changes in the other data set.

We conclude that the two data sets essentially agree on all prices reported.

### **A.1.3 Check 3: Quotes vs. actual trades**

As a third quality check, we compare the quotes from our main dataset to the prices of actual transactions reported by the Depository Trust & Clearing Corporation (DTCC), which has collected data on all trades of variance swaps in the US since 2013.<sup>2</sup>

Appendix Figure A.1 shows the distribution of the percentage difference between our quotes and the transaction prices for different maturity baskets. Quotes and transaction prices are in most cases very close, with the median absolute percentage difference across all maturities approximately 1 percent. We conclude that our quotes reflect the prices at which transactions occur with a high degree of accuracy.

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<sup>2</sup>DTCC was the only swap data repository registered under the Dodd–Frank act to collect data on variance swaps in 2013. The Dodd–Frank act requires that all swaps be reported to a registered data repository.

#### **A.1.4 Check 4: Variance Swaps vs. VIX vs. VIX Futures**

As discussed in the paper, in addition to the two datasets on variance swaps we also observe a term structure of synthetic variance swap prices (VIX) up to 12 months maturity, and of VIX futures (which are exchange-traded contracts) up to 6 months maturity. The three data sets come from entirely independent sources: variance swaps are traded over the counter, the VIX is constructed using options, and VIX futures are exchange traded. These additional data sources are affected by different liquidity and trading setups relative to variance swaps, and if those were particularly important in this market one would expect to find large deviations between these markets and the variance swap market. Instead, we show now that the three markets move essentially together at all maturities.

##### **A.1.4.1 Variance Swaps vs. the VIX**

The correlation of variance swap prices with the VIX at corresponding maturities is above 0.99 at all maturities. The correlation of price changes is more noisy, but still 0.96 on average across maturities and never below 0.94. None of the price changes in one dataset helps in predicting price changes in another dataset, indicating that no dataset is reacting to information later than the other.

##### **A.1.4.2 Variance Swaps vs. VIX futures**

Since VIX futures are claims to the VIX in  $n$  periods, they are equivalent to forward variance claims with maturity  $n + 1$  (apart of a small convexity adjustment due to the fact that the payoff of VIX futures is expressed in volatility, not variance, units). The correlation of prices is on average 0.993, and always at least 0.992. The correlation of price changes is 0.98 on average, and always at least 0.94. No significant predictive relation exists between price changes from the two sources, except that variance swaps predict VIX futures price changes with a t-stat of 2.08 in only one case (one significant case out of the many predictive tests we ran is not an indication that indeed our variance swap prices data is better than the VIX futures data, as it is likely only due to noise).

All of our tests of data quality suggest that the variance swap data is high quality, and displays no indication of bad reporting, stale prices, or differential price behavior due to different liquidity, compared to the other data sources.

## A.2 Synthetic variance swap prices

We construct option-based synthetic variance claims for maturity  $n$ ,  $VIX_{n,t}^2$  using the methods described by the CBOE (2009) in their construction of the VIX index, using data from Optionmetrics covering the period 1996 to 2012.

In particular, we construct  $VIX_{n,t}^2$  for maturity  $n$  on date  $t$  as

$$VIX_{n,t}^2 \equiv \frac{2}{n} \sum_i \frac{\Delta K_i}{K_i^2} \exp(-nR_{n,t}) P(K_i) \quad (\text{A.1})$$

where  $i$  indexes options;  $K_i$  is the strike price of option  $i$ ;  $\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}$  unless  $i$  is the first or last option used, in which case  $\Delta K_i$  is just the difference in strikes between  $K_i$  and its neighbor;  $R_{n,t}$  is the  $n$ -day forward yield (from Fama and Bliss (1987)); and  $P(K_i)$  is the midpoint of the bid-ask spread for the out-of-the-money option with strike  $K_i$ . The summation uses all options available with a maturity of  $n$  days. We deviate slightly here from the CBOE, which drops certain options with strikes very far from the current spot. For each  $t$  and  $n$ , we require the presence of at least 4 out-of-the-money calls and puts. We create  $VIX_{n,t}^2$  for all monthly maturities by interpolating between available option maturities, using the same techniques as the CBOE.

Under the assumption that the price of the underlying follows a diffusion (i.e. does not jump), it is the case that

$$VIX_{n,t}^2 = E_t^Q \left[ \int_t^{t+n} \sigma_j^2 dj \right] \quad (\text{A.2})$$

$$\approx E_t^Q \left[ \sum_{j=1}^n RV_{t+j} \right] \quad (\text{A.3})$$

where  $\sigma_t^2$  is the instantaneous volatility at time  $t$ .<sup>3</sup> The second line simply notes that the integrated volatility is approximately equal to the sum of squared daily returns (where the quality of the approximation improves as the sampling interval becomes shorter). In other words, when the underlying follows a diffusion,  $VIX_{n,t}^2$  corresponds to the price of an idealized variance swap where the squared returns are calculated at arbitrarily high frequency.

Given  $VIX_{n,t}^2$  prices at the monthly intervals, we also construct forward variance claims,  $VIXF_{n,t}$  as

$$VIXF_{n,t} = VIX_{n,t}^2 - VIX_{n-1,t}^2 \quad (\text{A.4})$$

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<sup>3</sup>See Carr and Wu (2009).

The forward prices obtained from options are very close to those obtained from variance swaps. Figure 6 in the text compares the two curves for the S&P 500. Figure A.2 compares the curves for the STOXX 50, FTSE 100 and DAX, and shows that the curves are similar for international markets as well (though in that case there is more noise).

### A.3 Decomposing the sources of variation of variance swap prices

In this section we investigate whether the variation in variance swap prices is primarily driven by changes in expected future volatility or changes in risk premia.<sup>4</sup>

Using the definition of forward variance claims, the following identity holds, for each maturity  $n$ :

$$F_t^n - F_t^0 = [E_t RV_{t+n} - RV_t] + \left[ -E_t \sum_{j=0}^{n-1} r_{t+1+j}^{n-j} \right] \quad (\text{A.5})$$

where  $r_t^n$  is the one-period return of the  $n$ -period forward claim. An increase in the  $n$ -period forward variance price must predict either an increase in future realized variance or lower future variance risk premia. Following Fama and Bliss (1987) and Campbell and Shiller (1991), we can then decompose the total variance of  $F_t^n - F_t^0$  into the component that predicts future  $RV$ , and the component that captures movements in risk premia. Note that just as in Fama and Bliss (1987) and Campbell and Shiller (1991), we perform this decomposition in changes (predicting the change in volatility rather than the level of volatility), because the previous empirical literature and our term-structure estimates of Section 4.3 highlight the presence of a very persistent factor in the volatility process.

The right side of Table A.1 shows this decomposition for different maturities between 1 month and 1 year. We see that most of the variation in variance swap prices can be attributed to movements in the expectation of future realized variance, not risk premia. In particular, at horizons of 3 to 12 months, essentially all the variation in prices is due to variation in expected volatility rather than variation in risk premia.

At the same time, we know from Table 2 and Figure 1 that prices of both short-term and long-term claims vary substantially: this indicates that the expectation of future realized variance changes dramatically over time. For example, the standard deviation of innovations in the 12-month forward claim is 17 percent per year. Given the finding in this section that the variance of  $F_t^{12}$  is driven entirely by changes in volatility expectations, we see that

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<sup>4</sup>A similar exercise was conducted by Mixon (2007), using S&P 500 options to predict future implied volatility.

investors' expectations of future volatility in fact vary substantially over time.

## A.4 Estimation of the no-arbitrage model

In this appendix, we provide more details on the estimation of the no-arbitrage model. We also report the estimated parameters and more detailed pricing results.

### A.4.1 Estimation Strategy

For estimation purposes, a standard and convenient practice of the term structure literature is to assume that some fixed-weights “portfolios” of VS prices are priced perfectly. These portfolios in turn allow one to invert for the latent states which are needed in the computation of the likelihood scores of the data.

A challenge in implementing this practice in our context is that for parts of our sample the set of available maturities may change from one observation to the next. In the later part of our sample, we use VS prices with maturities up to 14 years, whereas the longest maturity for the earlier sample is only two years.

To tackle this issue, we maintain the assumption from the term structure literature that the current term structure of the VS prices perfectly reveals the current values of states. Nonetheless, we depart from the standard term structure practice by using some time-varying-weights “portfolios” of VS prices in identifying the states at each point in time. The portfolios weights are determined in a way to optimally accommodate different sets of maturities at each point in the sample.

To begin, let  $D_t$  denote the vector of observed data obtained by stacking up the vector of VS prices on top of the realized variance  $RV_t$ . Because VS prices are affine in states, we can write:

$$D_t = A + BX_t. \tag{A.6}$$

All entries of the last column of  $B$ , except for the last row, are zeros because VS prices are only dependent on  $s_t^2$  and  $l_t^2$ . In addition, since the last entry of  $D_t$  corresponds to  $RV_t$ , the last row of  $A$  is 0 and the last row of  $B$  is (0,0,1). Keep in mind that the length of  $D_t$  can vary from time to time due to different maturity sets for the observed VS data.

We assume that  $D_t$  is observed with iid errors:

$$D_t^o = D_t + e_t \tag{A.7}$$

where  $D_t^o$  denotes the observed counterpart of  $D_t$ . Adopting a standard practice in the

term structure literature, we assume that the observational errors for the VS prices are uncorrelated and have one common variance  $\sigma_e^2$ . Because observed  $RV_t$  is used in practice to determine payoffs to VS contracts, it is natural to assume that  $RV_t$  is observed without errors. Combined, if a number of  $J$  VS prices are observed at time  $t$ , the covariance matrix of  $e_t$ ,  $\Sigma_e$ , takes the following form:

$$\Sigma_e = \begin{pmatrix} I_J \sigma_e^2 & 0_{J \times 1} \\ 0_{1 \times J} & 0 \end{pmatrix}, \quad (\text{A.8})$$

where  $I_J$  denotes the identity matrix of size  $J$ .

We now explain how we can recursively identify the states. Assume that we already know  $X_t$ . Now imagine projecting  $X_{t+1}$  on  $D_{t+1}^o$  conditioning on all information up to time  $t$ . Our assumption (borrowed from the term structure literature) that the current term structure of the VS prices perfectly reveals the current values of the states implies that the fit of this regression is perfect. In other the words, the predicted component of this regression, upon observing  $D_{t+1}^o$ ,

$$E_t(X_{t+1}) + cov_t(X_{t+1}, D_{t+1}^o) var_t(D_{t+1}^o)^{-1} (D_{t+1}^o - E_t(D_{t+1}^o)), \quad (\text{A.9})$$

must give us the values for the states at time  $t + 1$ :  $X_{t+1}$ . All the quantities needed to implement (A.9) are known given  $X_t$ . Specifically,  $E_t(X_{t+1}) = K_0 + K_1 X_t$  and  $var_t(D_{t+1}^o) = V_t(X_{t+1}) + \Sigma_e$  where  $V_t(X_{t+1})$  is computed according to each of our three specifications for the covariance.  $E_t(D_{t+1}^o)$  is given by  $A + B E_t(X_{t+1})$ . And  $cov_t(X_{t+1}, D_{t+1}^o)$  is given by  $V_t(X_{t+1}) B'$ . Clearly, the calculation in (A.9) can be carried out recursively to determine the values of  $X_t$  for the entire sample.

Our approach is very similar to a Kalman filtering procedure apart from the simplifying assumption that  $D_{t+1}^o$  fully reveals  $X_{t+1}$ . That is,  $V_t(X_{t+1} | D_{t+1}^o) \equiv 0$ . In a term structure context, Joslin, Le, and Singleton (2013) show that this assumption allows for convenient estimation, yet delivers typically highly accurate estimates.

We can view (A.9) as some “portfolios” of the observed data  $D_{t+1}^o$  with the weights given by:  $cov_t(X_{t+1}, D_{t+1}^o) var_t(D_{t+1}^o)^{-1}$ . As a comparison, whereas the term structure literature typically choose, prior to estimation, a fixed weight matrix corresponding to the lower principal components of the observed data, we do not have to specify *ex ante* any loading matrix. Our approach determines a loading matrix that optimally extracts information from the observed data and, furthermore, can accommodate data with varying lengths.

As a byproduct of the above calculations, we have available the conditional means and

variances of the observed data:  $E_t(D_{t+1}^o)$  and  $V_t(D_{t+1}^o)$ . These quantities allow us to compute the log QML likelihood score of the observed data as (ignoring constants):

$$\mathcal{L} = \sum_t -\frac{1}{2} \|\text{var}_t(D_{t+1}^o)^{-1/2}(D_{t+1}^o - E_t(D_{t+1}^o))\|_2^2 - \frac{1}{2} \log |\text{var}_t(D_{t+1}^o)|. \quad (\text{A.10})$$

Estimates of parameters are obtained by maximizing  $\mathcal{L}$ . Once the estimates are obtained, we convert the above QML problem into a GMM setup and compute robust standard errors using a Newey West matrix of covariance.

## A.4.2 Alternative variance specifications

### Constant variance structure

In the first alternative specification, we let  $V_t(X_{t+1})$  be a constant matrix  $\Sigma_0$ . Since both  $E_t(X_{t+1})$  and  $E_t^Q(X_{t+1})$  are linear in  $X_t$ ,  $\Lambda_t$  is also linear in  $X_t$ . We refer to this as the CV (for constant variance) specification.

### Flexible structure

It is important to note that the specifications of  $\Lambda_t$  in (12) and  $V_t(X_{t+1})$  in (13) introduce very tight restrictions on the difference:  $E_t(X_{t+1}) - E_t^Q(X_{t+1})$ . Simple algebra shows that the first entry of the product  $V_t(X_{t+1})^{1/2}\Lambda_t$  is simply a scaled version of  $s_t^2$ . This means that the dependence of  $E_t(X_{t+1})$  and  $E_t^Q(X_{t+1})$  on  $l_t^2$  and  $RV_t$  and a constant must be exactly canceled out across measures. Similar arguments lead to the following restrictions on the condition mean equation:

$$E_t \begin{pmatrix} s_{t+1}^2 \\ l_{t+1}^2 \\ RV_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ v_l^Q \\ 0 \end{pmatrix}}_{K_0} + \underbrace{\begin{pmatrix} \rho_s & 1 - \rho_s^Q & 0 \\ 0 & \rho_l & 0 \\ \rho_{s,RV} & 0 & 0 \end{pmatrix}}_{K_1} \begin{pmatrix} s_t^2 \\ l_t^2 \\ RV_t \end{pmatrix}. \quad (\text{A.11})$$

So in the CIR specification, the conditional mean equation only requires three extra degrees of freedom in:  $\rho_s$ ,  $\rho_l$ , and  $\rho_{s,RV}$ . The remaining entries to  $K_0$  and  $K_1$  are tied to their risk-neutral counter parts. By contrast, all entries of  $K_0$  and  $K_1$  are free parameters in the CV specification. Whereas CV offers more flexibility in matching the time series dynamics of  $X_t$ , the parsimony of CIR, if well specified, can potentially lead to stronger identification. However, this parsimony of the CIR specification, if mis-specified, can be

restrictive. For example, in the CIR specification,  $RV_t$  is not allowed to play any role in forecasting  $X$ .

In the flexible specification, we would like to combine the advantages of the CV specification in matching the time series dynamics of  $X$  and the advantages of the CIR specification in modeling time-varying volatilities.

For time-varying volatility, we adopt the following parsimonious structure:

$$V_t(X_{t+1}) = \Sigma_1 s_t^2, \quad (\text{A.12})$$

where  $\Sigma_1$  is a fully flexible positive definite matrix. This choice allows for non-zero covariances among all elements of  $X$ .

The market prices of risks are given by:

$$\Lambda_t = V_t(X_{t+1})^{-1/2}(E_t(X_{t+1}) - E_t^Q(X_{t+1})), \quad (\text{A.13})$$

where the superscript  $^{1/2}$  indicates a lower triangular Cholesky decomposition. Importantly, we do not require the market prices of risks to be linear, or of any particular form. This is in stark contrast to the CIR specification which requires the market prices of risks to be scaled versions of states. As a result of this relaxation, no restrictions on the conditional means dynamics are necessary. Aside from the non-negativity constraints, the parameters  $K_0, K_1$  that govern  $E_t(X_{t+1}) = K_0 + K_1 X_t$  are completely free, just as in the CV specification. In particular,  $RV_t$  is allowed to forecast  $X$  and thus can be important in determining risk premiums. We label this specification as the FLEX specification (for its flexible structure).

### A.4.3 Additional estimation results

Table A.2 reports the risk neutral parameters of our no arbitrage models:  $\rho_s^Q$ ,  $\rho_l^Q$ , and  $v_l^Q$ . As expected, these parameters are very strongly identified thanks to the rich cross-section of VS prices used in the estimation. Recall that our risk-neutral construction is identical for all three of our model specifications. As a result, estimates of risk neutral parameters are nearly invariant across different model specifications.

The effect of including the crisis is that the estimates for  $\rho_s^Q$  and  $v_l^Q$  are higher, whereas the estimate of  $\rho_l^Q$  is the same. A higher  $v_l^Q$  is necessary to fit a higher average VS curve. A higher estimate for  $\rho_s^Q$  implies that risk-neutral investors perceive the short-run factor  $s_t^2$  as more (risk-neutrally) persistent. In other words, movements of the one-month VS price ( $s_t^2$ ) will affect prices of VS contracts of much longer maturities.

We report the time series parameters –  $K_0, K_1$ , and other parameters that govern the

conditional variance  $V_t(X_{t+1})$  – for the CV, CIR, and FLEX specifications in Tables A.3, A.4, and A.5, respectively.

The values of annualized steady-state risk prices implied by each model specification are reported in A.6. Regardless of how the quantity of risk ( $V_t(X_{t+1})$ ) is modeled, RV-risk is always very significantly negatively priced. The point estimates and standard errors are similar across the various specifications for the variance process (and hence the physical dynamics), emphasizing the results of our findings. The table also shows that the results are similar regardless of whether the financial crisis is included in the estimation sample. Since the financial crisis was a period when the returns on variance swaps were extraordinarily high, excluding it from the data causes to estimate risk prices that are even more negative than in the full sample.

## A.5 Calibration and simulation of the models

This section gives the details of the three models analyzed in the main text.

### A.5.1 Long-run Risk (Drechsler and Yaron)

Our calibration is identical to that of Drechsler and Yaron (DY; 2011), so we refer the reader to the paper for a full description of the model. We have confirmed that our simulation matches the moments reported in DY (tables 6, 7, and 8, in particular).

### A.5.2 Time-varying disasters (Wachter)

The key equations driving the economy are

$$\Delta c_t = \mu_{\Delta c} + \sigma_{\Delta c} \varepsilon_{\Delta c,t} + J_{\Delta c,t} \tag{A.14}$$

$$F_t = (1 - \rho_F) \mu_F + \rho_F F_{t-1} + \sigma_F \sqrt{F_{t-1}} \varepsilon_{F,t} \tag{A.15}$$

$$\Delta d_t = \lambda \Delta c_t \tag{A.16}$$

where  $\varepsilon_{\Delta c,t}$  and  $\varepsilon_{F,t}$  are standard normal innovations. This is a discrete-time version of Wachter’s setup, and it converges to her model as the length of a time period approaches zero. The model is calibrated at the monthly frequency. Conditional on a disaster occurring,  $J_{\Delta c,t} \sim N(-0.3, 0.15^2)$ . The number of disasters that occurs in each period is a Poisson variable with intensity  $F_t$ . The other parameters are calibrated as:

Parameter	Value	Parameter	Value
$\mu_{\Delta c}$	0.02/12	$\sigma_{\Delta c}$	0.02/sqrt(12)
$\sigma_F$	0.0075	$\lambda$	2.8
$\rho_F$	$0.87^{1/12}$	$\mu_F$	0.017/12

In the analytic solution to the model, the price/dividend ratio for a levered consumption claim takes the form  $pd_t = z_0 + z_1 F_t$ . The Campbell–Shiller approximation to the return (which becomes arbitrarily accurate as the length of a time interval shrinks) is

$$r_{t+1} = \theta pd_{t+1} + \Delta d_{t+1} - pd_t \quad (\text{A.17})$$

$$= \theta pd_{t+1} + \lambda \Delta c_{t+1} - pd_t \quad (\text{A.18})$$

It is straightforward to show analytically (the derivation is available on request) that

$$pd_t = z_0 + z_1 F_t \quad (\text{A.19})$$

In the absence of a disaster, we treat the shocks to consumption and the disaster probability as though they come from a diffusion, so that the realized variance is  $\theta^2 z_0^2 \sigma_F^2 F_{t-1} + \lambda \sigma_{\Delta c}^2$ . When a disaster occurs, we assume that the largest daily decline in the value of the stock market is 5 percent. So, for example, a 30-percent decline would be spread over 6 days. The results are largely unaffected by the particular value assumed. The realized variance when a disaster occurs is then  $\theta^2 z_0^2 \sigma_F^2 F_{t-1} + \lambda \sigma_{\Delta c}^2 - (0.05) J_{\Delta c, t}$  (assuming  $J_{\Delta c, t} \leq 0$ ).

The model is solved analytically using methods similar to those in DY. Specifically, household utility,  $v_t$ , is

$$v_t = (1 - \beta) c_t + \frac{\beta}{1 - \alpha} \log E_t [\exp((1 - \alpha) v_{t+1})] \quad (\text{A.20})$$

$\alpha$  is set to 3.6 and  $\beta = 0.98^{1/12}$ . The recursion can be solved because the cumulant-generating function for a poisson mixture of normals (the distribution of  $J_{\Delta c}$ ) is known analytically (again, see DY).

The pricing kernel is

$$M_{t+1} = \beta \exp(-\Delta c_{t+1}) \frac{\exp((1 - \alpha) v_{t+1})}{E_t [\exp((1 - \alpha) v_{t+1})]} \quad (\text{A.21})$$

Asset prices, including those for claims on realized variance in the future, then follow immediately from the solution of the lifetime utility function and cumulant-generating function for  $J_{\Delta c}$ . The full derivation and replication code is available upon request.

We attempted to keep the calibration as close as possible to Wachter's. The two differences are that we increase risk aversion somewhat in order to try to generate a larger 1-month variance risk premium and that we use a normal distribution for the disasters rather than the empirical distribution used by Wachter (to allow us to obtain analytic results). It is important to note that risk aversion cannot be increased past 3.6 because the model no longer has a solution.

As with the DY model, we checked that the moments generated by our solution to the model match those reported by Wachter. We confirm that our results are highly similar, in particular for her tables 2 and 3, though not identical since we use slightly different risk aversion and a different disaster distribution.

### A.5.3 Disasters with time-varying recovery (Gabaix)

$$\Delta c_t = \mu_{\Delta c} + \varepsilon_{\Delta c,t} + J_{\Delta c,t} \tag{A.22}$$

$$L_t = (1 - \rho_L) \bar{L} + \rho_L L_{t-1} + \varepsilon_{L,t} \tag{A.23}$$

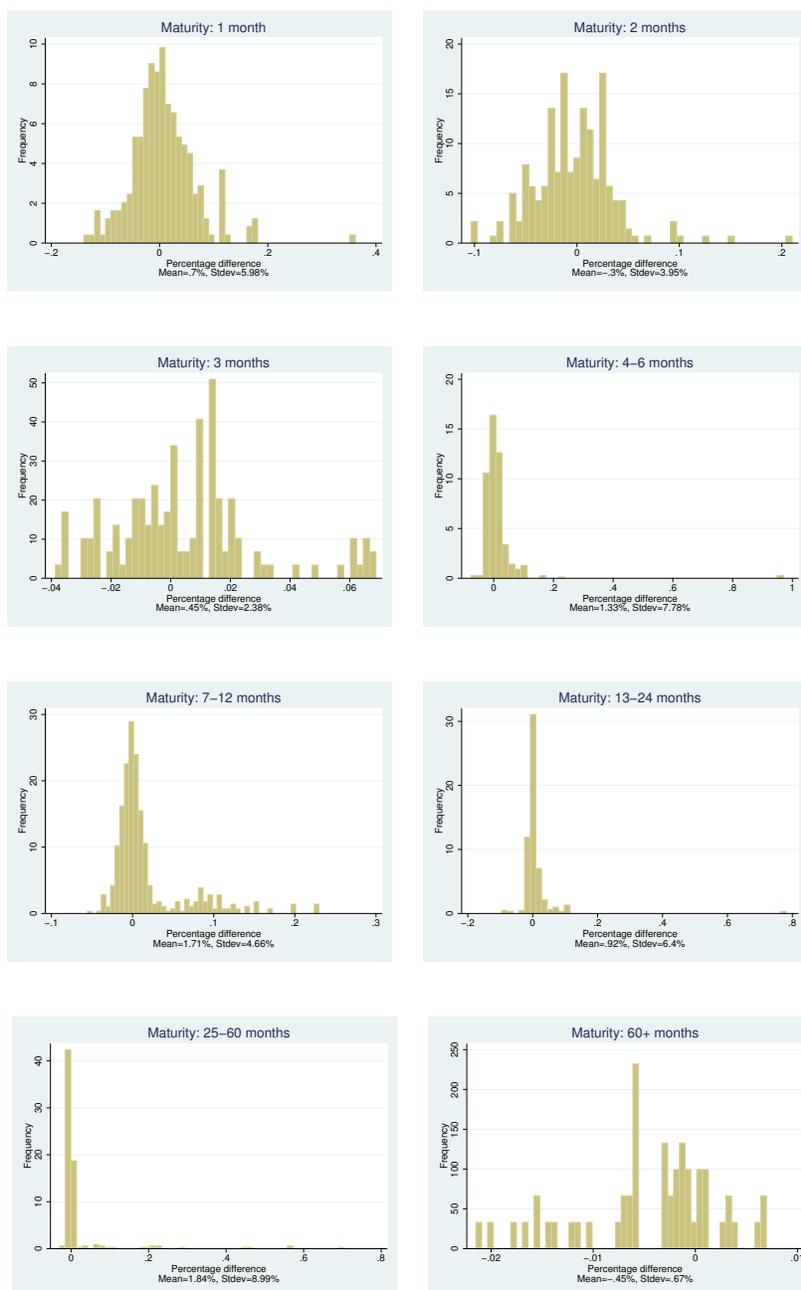
$$\Delta d_t = \lambda \varepsilon_{\Delta c,t} - L \times 1 \{J_{\Delta c,t} \neq 0\} \tag{A.24}$$

The model is calibrated at the monthly frequency. Conditional on a disaster occurring,  $J_{\Delta c,t} \sim N(-0.3, 0.15^2)$ . The probability of a disaster in any period is 0.01/12. The other parameters are calibrated as:

Parameter	Value
$\mu_{\Delta c}$	0.1/12
$\text{stdev}(\varepsilon_{\Delta c})$	$0.02/\sqrt{12}$
$\rho_L$	$0.87^{1/12}$
$\bar{L}$	0.5
$\text{stdev}(\varepsilon_L)$	0.04
$\lambda$	5

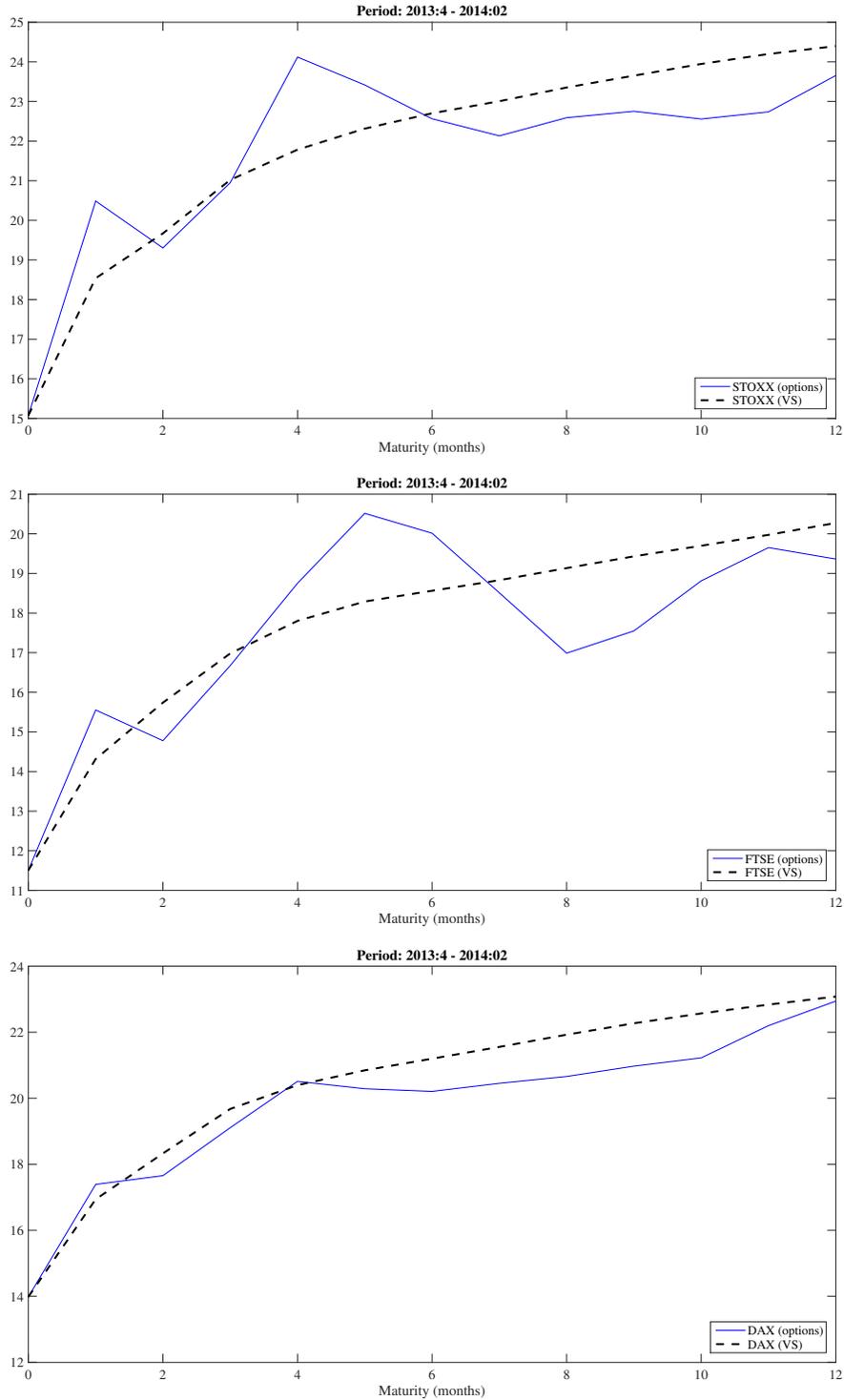
Agents have power utility with a coefficient of relative risk aversion of 7 and a time discount factor of  $0.96^{1/12}$ .

Figure A.1: Quotes vs. transaction prices



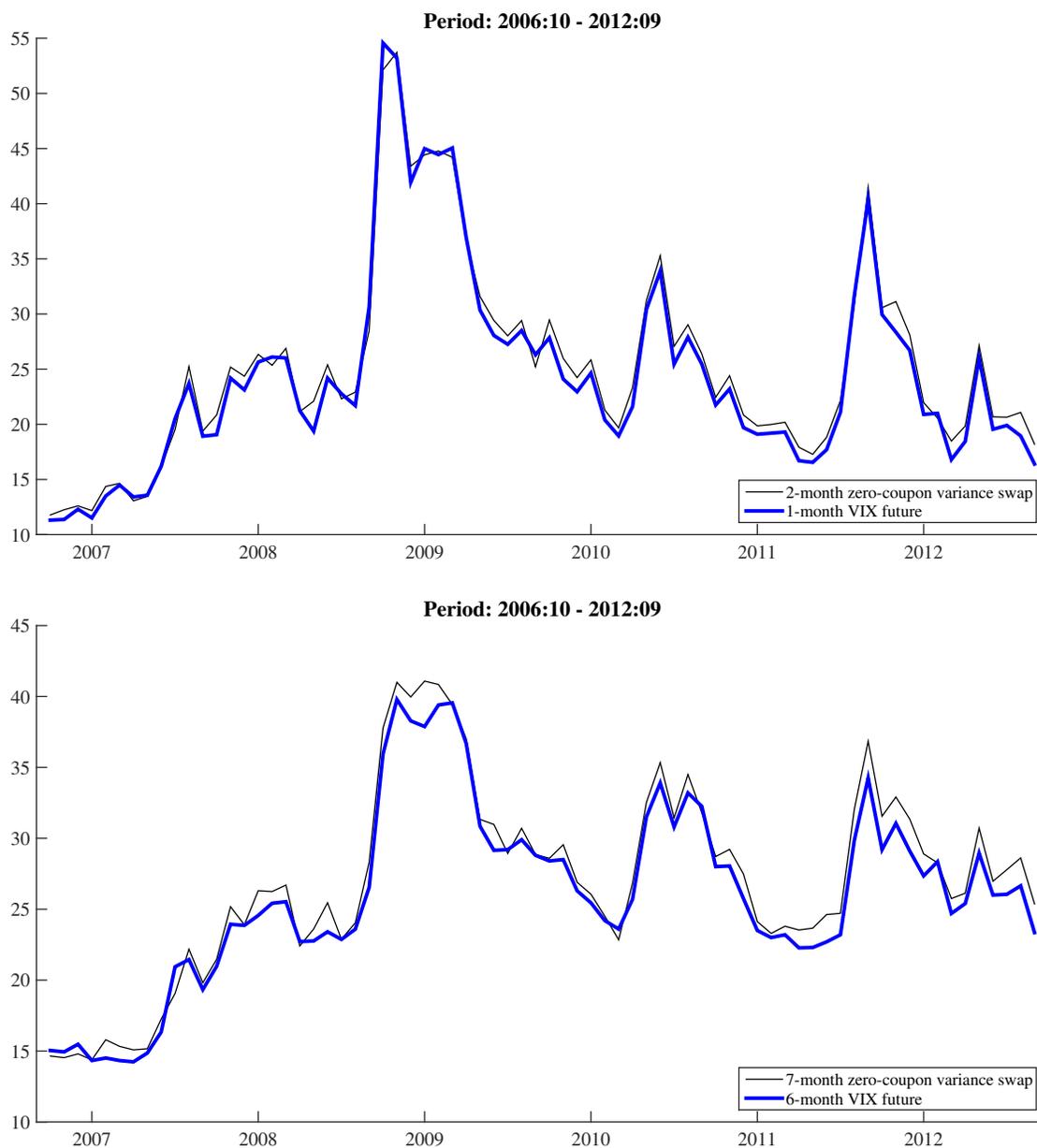
**Note:** The figure shows the distribution of percentage difference between variance swap price quotes and actual transaction prices, computed as  $(\text{transaction price} - \text{quote})/\text{quote}$ . The quotes are our main sample, while transaction prices are obtained from the DTCC and begin in 2013. Each panel shows the histogram for a different bucket of maturity of the variance swap contracts.

Figure A.2: International forward variance claim prices from options and variance swaps



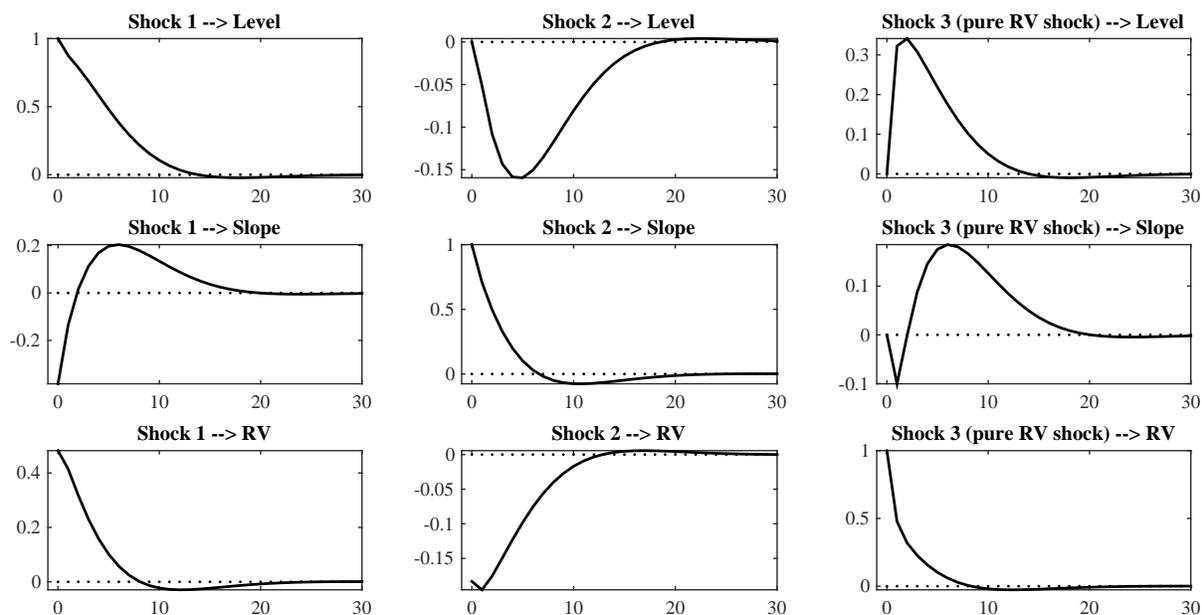
**Note:** In each panel (corresponding to STOXX 50, FTSE 100 and DAX), the solid line plots average prices of forward variance claims calculated using the formula for the VIX index and data on international option prices from Optionmetrics. The dotted line plots the average prices of the same claims computed from international variance swaps. The sample covers the period 2013:4-2014:2.

Figure A.3: VIX futures vs. forward variance swap prices



**Note:** the top panel plots the time series of the 2-month forward variance swap and the 1-month VIX future price from the CME, in annualized volatility terms. The bottom panel plots the time series of the 7-month forward variance swap and the 6-month VIX future price from the CME, in annualized volatility terms. The sample covers the period 2006:10-2012:9.

Figure A.4: Impulse response functions of level, slope and RV



**Note:** Each panel plots the response of one of the variables in the VAR (level, slope, and RV) to one of the three orthogonalized shocks. The shocks are orthogonalized with a Cholesky factorization with the ordering level-slope-RV.

Table A.1: Predictive regressions

Horizon (months)	Predictor: Slope and Curvature		Predictor: $F_t^n - F_t^0$					
	Dep var: $RV_{t+n} - RV_t$	$R^2$	Dep var: $RV_{t+n} - RV_t$	$R^2$	coeff.	Dep var: $-\sum_{j=0}^{n-1} r_{t+1+j}^{n-j}$	$R^2$	coeff.
1	0.08		0.17	0.63***	(0.22)	0.07	0.37*	(0.22)
2	0.22		0.18	0.71***	(0.17)	0.04	0.29*	(0.17)
3	0.31		0.24	0.83***	(0.14)	0.01	0.17	(0.14)
6	0.36		0.34	0.95***	(0.09)	0.001	0.05	(0.09)
12	0.38		0.41	1.10***	(0.09)	0.001	-0.10	(0.09)

**Note:** Results of regressions forecasting changes in realized variance. The left side reports the  $R^2$  of a regression of changes in realized volatility between month  $t$  and month  $t + n$  on the level and the slope at time  $t$ . The right side reports the coefficients of univariate regressions of changes in realized volatility (left column) and returns to volatility claims from  $t$  to  $t + n$  (right column) on the difference between the forward prices of maturity  $n$  ( $F_t^n$ ) and realized volatility ( $F_t^0$ ) at time  $t$ . \* indicates significance at the 10-percent level, \*\* the 5-percent level, and \*\*\* the 1-percent level.

Table A.2: Risk neutral parameters

Specifications		1996:2007			1996:2013		
		$\rho_s^Q$	$\rho_t^Q$	$v_t^Q$	$\rho_s^Q$	$\rho_t^Q$	$v_t^Q$
CV	Est.	0.66***	0.99***	0.65***	0.83***	0.99***	1.01**
	Stderr.	0.06	0.00	0.14	0.07	0.01	0.48
CIR	Est.	0.68***	0.99***	0.65***	0.84***	0.99***	0.99**
	Stderr.	0.05	0.00	0.14	0.06	0.01	0.44
FLEX	Est.	0.66***	0.99***	0.64***	0.83***	0.99***	1.01**
	Stderr.	0.06	0.00	0.14	0.07	0.01	0.47

**Note:** The table reports the risk-neutral estimated dynamics of the term structure model, for the three specifications CV, CIR and FLEX, and separately for the full sample (1996-2013) and the pre-crisis sample (1996-2007).

Table A.3: Time series parameter estimates for the CV specification

Sample		$K_0$	$K_1$			$\Sigma^*$		
1996:2007	Est.	6.49***	0.65***	0.14**	0.03	15.42***	0	0
		3.36***	0.08	0.83***	0.02	3.87***	5.25***	0
		1.98	0.56***	0	0.12	10.57***	-2.62	14.08***
	Std. err.	2.42	0.08	0.07	0.12	3.23	-	-
		1.12	0.10	0.08	0.07	0.73	0.73	-
		2.11	0.07	-	0.10	2.36	2.63	2.07
1996:2013	Est.	8.99***	0.56***	0.08	0.19**	20.59***	0	0
		2.43***	0	0.91***	0.06***	3.26	7.04***	0
		5.67*	0.20	0	0.57***	30.00**	-2.14	25.49***
	Std. err.	2.48	0.13	0.05	0.09	5.07	-	-
		0.90	-	0.03	0.01	2.09	0.97	-
		2.92	0.13	-	0.15	13.38	2.94	4.23

**Note:** The table reports the time-series parameter estimates for the CV specification.  $\Sigma^*$  is the lower triangular Cholesky decomposition of  $\Sigma_0$ . For admissibility,  $K_0$  and  $K_1$  are constrained to be non-negative. Those entries for which the non-negativity constraint is binding are set to zero and thus standard errors are not provided. The table reports the estimates separately for the full sample (1996-2013) and the pre-crisis sample (1996-2007).

Table A.4: Time series parameter estimates for the CIR specification

Sample		$K_0$	$K_1$			$\Sigma^*$		
1996:2007	Est.	0	0.66***	0.32***	0	5.16***	0	3.53***
		0.65***	0	0.99***	0	0	0.98***	0
		0	0.69***	0	0	3.53***	0	6.74***
	Std. err.	-	0.07	0.05	-	1.82	-	1.18
		0.14	-	0.00	-	-	0.23	-
		-	0.01	-	-	1.18	-	1.69
1996:2013	Est.	0	0.82***	0.16**	0	7.17**	0	9.42*
		0.99**	0	0.98***	0	0	1.42**	0
		0	0.75***	0	0	9.42*	0	23.56*
	Std. err.	-	0.04	0.06	-	2.84	-	5.42
		0.44	-	0.00	-	-	0.58	-
		-	0.09	-	-	5.42	-	12.88

**Note:** The table reports the time-series parameter estimates for the CIR specification. The diagonal elements of  $\Sigma^*$  correspond to the variance parameters  $\sigma_s^2$ ,  $\sigma_l^2$ , and  $\sigma_{RV}^2$ . The (3,1) entry of  $\Sigma^*$  correspond to the covariance parameter  $\sigma_{s,RV}$ . The table reports the estimates separately for the full sample (1996-2013) and the pre-crisis sample (1996-2007).

Table A.5: Time series parameter estimates for the FLEX specification

Sample		$K_0$	$K_1$			$\Sigma^*$		
1996:2007	Est.	2.31	0.74***	0.19**	0.01	2.26***	0	0
		2.18***	0.09*	0.87***	0	0.58***	0.78***	0
		0.11	0.60***	0	0.13	1.56***	-0.24	2.04***
	Std. err.	1.49	0.08	0.08	0.08	0.42	-	-
		0.81	0.05	0.06	-	0.08	0.08	-
		0.75	0.08	-	0.09	0.29	0.31	0.27
1996:2013	Est.	4.54***	0.68***	0.08**	0.19**	2.59***	0	0
		1.37**	0	0.93***	0.07***	0.44*	1.09***	0
		0.67***	0.36***	0	0.51**	3.32***	-0.34	3.20***
	Std. err.	1.30	0.08	0.04	0.08	0.48	-	-
		0.55	-	0.02	0.02	0.23	0.12	-
		0.05	0.11	-	0.20	1.24	0.43	0.48

**Note:** The table reports the time-series parameter estimates for the FLEX specification.  $\Sigma^*$  is the lower triangular Cholesky decomposition of  $\Sigma_0$ . For admissibility,  $K_0$  and  $K_1$  are constrained to be non-negative. Those entries for which the non-negativity constraint is binding are set to zero and thus standard errors are not provided. The table reports the estimates separately for the full sample (1996-2013) and the pre-crisis sample (1996-2007).

Table A.6: Annualized steady state risk prices, all specifications

Specifications	Sources of risks	1996:2007		1996:2013	
		Estimates	Standard Error	Estimates	Standard Error
CV	$s_t^2$ -risk	-0.23	0.22	-0.08	0.14
	$l_t^2$ -risk	0.05	0.18	-0.21	0.14
	RV-risk	-2.78***	0.65	-1.44***	0.43
CIR	$s_t^2$ -risk	-0.18	0.36	-0.11	0.31
	$l_t^2$ -risk	0.04	0.14	-0.18	0.12
	RV-risk	-3.92***	0.91	-1.70***	0.48
FLEX	$s_t^2$ -risk	-0.23	0.30	-0.14	0.16
	$l_t^2$ -risk	0.06	0.22	-0.17	0.19
	RV-risk	-3.17***	0.78	-1.69**	0.73

**Note:** Estimates of steady-state risk prices from the no-arbitrage model. Risk prices are annualized by multiplying by  $\sqrt{12}$ . \*\*\* denotes significance at the 1-percent level. Results are reported for the full sample (1996-2013), and restricted to 1996-2007.