On the effects of restricting short-term investment*

Nicolas Crouzet, Ian Dew-Becker, and Charles G. Nathanson

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Abstract

We study the effects of policies proposed for addressing "short-termism" in financial markets. We examine a noisy rational expectations model in which the exposures of investors and their information about fundamentals endogenously vary across horizons. In this environment, taxing or outlawing short-term investment has zero effect on the information in prices about long-term fundamentals. However, such a policy reduces the profits and utility of short- and long-term investors. Limiting the release of short-term information helps long-term investors (an objective of some policymakers) at the expense of short-term investors, but it also makes prices less informative and increases costs of speculation.

For decades economists and policymakers have expressed concern about the potentially negative effects of "short-termism" in financial markets. Research has argued that short-term investors may increase the volatility and reduce the informativeness of asset prices (Froot, Scharfstein, and Stein (1992)), exacerbate fire sales and crashes (Cella, Ellul, and Giannetti (2013)), inefficiently incentivize managers to focus on short-term projects (Shleifer and Vishny (1990)), or reduce incentives of other investors to acquire information (Baldauf and Mollner (2017); Weller (2017)), making prices less informative overall.

Those who take the view that short-termism is bad for financial markets or the economy as a whole have proposed a broad array of policies to encourage long-term investment. One of the

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oldest proposals is the tax on transactions of Tobin (1978).[1] Some policies directly depend on holding periods, such as US tax treatment of capital gains and dividends, the SEC's most recent proxy access rules, the proposed Long-Term Stock Exchange, linking corporate voting rights to tenure, and the proposal of Bolton and Samama (2013) for corporations to explicitly reward long-term investors.[2] Budish, Cramton, and Shim (2015) propose to eliminate trade at the very highest frequencies by shifting markets from continuous operation to frequent batch auctions, and there have also been proposals to limit or eliminate quarterly financial reports and earnings guidance in the US, following similar changes in the UK, e.g. by Dimon and Buffett (2018).[3] A number of these policies were endorsed in a letter from 2009 signed by leaders in business, finance, and law.[4]

This paper theoretically evaluates the effect of policies targeting short-termism on price informativeness and investor outcomes. Unlike the previous literature, we consider a simple and very general setting with investors who are ex ante identical and then may endogenously specialize into different horizons. While there is some recent work on the consequences of various limits on information gathering ability and there have been empirical analyses of high-frequency traders, we are not aware of any other work that directly studies the effects of restrictions on short- and long-term strategies on price informativeness and investor profits in a general setting.[5]

The model we study is designed to be as simple and general as possible. Two key features that it must have are that investors choose among investment strategies at different horizons, and that they choose how much information to acquire about fundamentals across horizons. We study a version of the noisy rational expectations model developed in Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016). Whereas that paper studies investment in a cross-section of assets, we argue here that investment policies over time can be thought of as a choice of exposures on many different future dates. Each of those dates represents a different "asset", and the returns on those assets

[1] See also Stiglitz (1989), Summers and Summers (1989), and Habermeier and Kirilenko (2003)

[2] See LTSE.org and Osipovich and Berman (2017).

[3] See also Nallareddy, Pozen, and Rajgopal (2016).

[4] "Overcoming Short-termism: A Call for a More Responsible Approach to Investment and Business Management", available at https://assets.aspeninstitute.org/content/uploads/files/content/docs/pubs/overcome_short_state0909_0.pdf. See also Stiglitz (2015).

[5] In much recent work, including Cartea and Penalva (2012), Baldauf and Mollner (2017) and Biais, Foucault, and Moinas (2015), high-frequency or short-term investors are somehow different from others, either in preferences or trading technologies. Those models are better suited to studying high-frequency trade specifically.

For recent analyses of limits on information gathering ability, see Banerjee and Green (2015), Goldstein and Yang (2015), Dávila and Parlatore (2016), and Farboodi and Veldkamp (2017).

will be correlated across dates.[6] The model in this paper is notable for allowing an arbitrarily long horizon (as opposed to two or three periods), with turnover at any frequency.

It is important to note that the model is not fully dynamic – all trade happens on date 0. It is well known that dynamic market equilibria are difficult or impossible to solve, and we do not contribute to that area.[7] The paper's focus is instead on the choice of short- versus long-term investment strategies and information acquisition. Short-term investors arise naturally in the model as agents whose exposures to fundamentals change relatively frequently across dates due to the type of information they have acquired.

There are a number potential reasons why a policymaker might want to regulate investment. As is common in the literature, those reasons are somewhat outside the model. For example, research often examines how policies affect price efficiency, even though the models studied do not generally imply that price efficiency raises welfare. There are at least three potential motivations. First, if price informativeness at long horizons is more important for economic decisions like physical investment, then long-term information acquisition might be encouraged. Second, policymakers might have a general bias toward long-term investors, perhaps because they are more likely to be people saving for retirement. Finally, one might think of the noise traders in the model as retail investors who make poor investment decisions driven by sentiment, or perhaps as uninformed speculators.[8] We use the model to examine how restrictions on investment policies affect price informativeness and the profits and utility of the various investors in order to help inform the policy debate. If the goal is to reduce mistakes or uninformed speculation, then one would ask how

[6]The paper uses a frequency transformation that allows the model to be solved by hand. For other related work on frequency transformations, see Bandi and Tamoni (2014), Bernhardt, Seiler, and Taub (2010), Chinco and Ye (2017), Chaudhuri and Lo (2016), Dew-Becker and Giglio (2016), and Kasa, Walker, and Whiteman (2013).

[7]The lack of dynamics means that the model is not suited for studying the relationship between investor horizon and bubbles, such as those of Blanchard (1979). There is work that has made substantial progress in solving the infinite regress problem, but those models assume that investors have only single-period objectives and they do not allow for a choice of information across horizons. See Makarov and Rytchkov (2012), Kasa, Walker, and Whiteman (2013), and Rondina and Walker (2017). Recent work also examines dynamic models with strategic trade (with similar restrictions regarding horizons), whereas here we study a fully competitive setting in which all investors are price takers – see Vayanos (1999, 2001), Ostrovsky (2012), Banerjee and Breon-Drisch (2016), Foucault, Hombert, and Rosu (2016), Du and Zhu (2017), and Dugast and Foucault (2017).

[8]One view is that policies aimed at short-termism are trying to reduce speculation, but that term is somewhat fuzzy. Sometimes speculators are simply investors with no fundamental hedging demand, in which case we would say that all the sophisticated investors in our model are speculators. Alternatively, speculators might be agents who invest based on signals about the demand of others, rather than about fundamentals. In the present setting, a signal about demand, after conditioning on prices, is directly informative about fundamentals, so there is little economic difference between the two here. We thus focus on motivations for addressing short-termism that have direct counterparts in the model.

to reduce the losses borne by noise traders and their effects on prices.

The paper examines a number of specific policies, including direct restrictions on investment strategies (which map to the batch auction mechanism of Budish, Cramton, and Shim (2015)), taxes on transactions, and taxing or subsidizing information acquisition. As to transaction taxes and investment restrictions, we show that when sophisticated agents are restricted from investing and trading at some frequency, prices become uninformative at that frequency. So if a policy were implemented saying that investors could no longer maintain positions for less than a month, variation in prices within the month would become uninformative for fundamentals, instead driven purely by liquidity demand. Intra-month price volatility and mean reversion would also rise. However, there is no spillover across horizons. A short-term restriction does not affect price informativeness or return volatility at longer horizons, so prices would remain informative at frequencies lower than a month. Transaction taxes also act as a restriction on short-term investment.

The next question is how investment restrictions affect investor outcomes. While it seems inevitable that a restriction on short-term investment would reduce the welfare of short-term investors, it is less obvious what would happen to long-term investors or noise traders. An increase in the number of short-term investors (e.g. due to a change in technology that makes short-term investment cheaper) turns out to make long-term investors worse off, essentially taking away some of the long-term investors' trading opportunities. But restricting short-term investment does not transfer profits back to long-term investors; instead it simply eliminates those profits, making both short- and long-term investors worse off.

In the context of the model, the way to tilt markets in favor of long-term investors – if that is one's goal – is to restrict the availability of short-term information. There have been numerous recent proposals to do just that, for example by limiting quarterly earnings guidance (e.g. Schacht et al. (2007), Pozen (2014), Dimon and Buffett (2018), and the Aspen Institute's report). In the UK, in fact, such reports are no longer mandatory for publicly traded companies. The model in this paper is well suited to analyze such policies, and we show that they are ideally targeted to shifting the equilibrium toward long-term investors, increasing their average profits and utility.

Finally, the paper examines the impact of the various policies on the profits of noise traders. Intuitively, the noise traders are constantly making mistakes, potentially affecting prices. There are two ways to protect them from those mistakes: stop them from trading, or reduce the losses they take on each trade. Stopping them from trading is in principle simple – just close asset markets – but then one loses the information contained in prices (along with any gains from trade). More interestingly, the paper shows that a better alternative is to subsidize or otherwise encourage information acquisition, which causes prices to become more informative and less responsive to noise trader (perhaps speculative) demand. Such a policy can eventually drive noise trader losses to zero, while simultaneously making prices more useful for economic decisions and reducing the excess volatility caused by noise trader speculation. However, and interestingly, it is the opposite of the policy that we showed helps the long-term investors.

Overall, then, we obtain three basic results about policies aimed at short-termism:

1. Restricting short-term investment affects short-run but not long-run price informativeness and return volatility.

2. Restricting short-term investment hurts both short- and long-term investors, but helps noise traders.

3. Taxing or restricting the availability of short-term information helps long-term investors, hurts short-term investors and noise traders, and reduces short-term price efficiency. Subsidizing information does the opposite.

On net, then, we would argue that subsidizing information acquisition is the most natural policy to address short-termism, as it both reduces speculative effects on prices and improves price efficiency. It does, however, come with costs to long-term investors. It is also the opposite of the recent proposals to reduce quarterly reporting.

The answers to the questions of how restrictions on trade affect price informativeness and welfare are not obvious ex ante. One view is that there might be some sort of separation across frequencies, so that restrictions in one realm do not affect outcomes in another. On the other hand, investors obviously interact – they trade with each other – so it would be surprising if policies targeting a particular type of investor did not act to benefit others. What we find is a mix of the two: market characteristics at high frequencies can affect the profits and utility of long-term investors – the model is not entirely separable across frequencies in that sense – but they do not affect low-frequency price informativeness. Furthermore, there is a tension between helping long-term investors, helping noise traders, and maintaining price informativeness. No single policy helps all the groups at the same time (due to a zero-sum aspect of the model) and policies that may be attractive to certain investors can come with negative side effects for agents outside the model – e.g. executives, or policymakers like the FOMC – who might make decisions based on asset prices.

The remainder of the paper is organized as follows. Sections 1 and 2 lay out the model and its solution. Section 3 examines the effects for price volatility and informativeness of restrictions on investment at different horizons, while section 4 examines the impacts of such policies on the profits and welfare of different investors. Section 4 also examines the impact of restrictions on information releases such as earnings announcements. Section 5 studies the effects of trading costs, and section 6 concludes.

1 The model

1.1 Market structure

Time is denoted by $t \in \{-1, 0, 1, ..., T\}$, with T even, and we will focus on cases in which T may be treated as large. There is a fundamentals process D_t, on which investors trade forward contracts, with realizations on all dates except -1 and 0. The time series is stacked into a vector $D \equiv [D_1, D_2, ..., D_T]'$ (versions of variables without time subscripts denote vectors) and is unconditionally distributed as

$$D \sim N(0, \Sigma_D). \tag{1}$$

For our benchmark results, we focus on the case where fundamentals are stationary. Appendix G shows that the results extend naturally to a case in which fundamentals are stationary in their growth rate, rather than their level. We discuss that case further below. Stationarity implies that Σ_D is constant along its diagonals, and we further assume that the eigenvalues of Σ_D are finite and bounded away from zero (which is satisfied by standard ARMA processes).

The biggest restriction imposed by the stationarity assumption (whether in levels or differences) is that we are assuming that the distribution of fundamentals is determined entirely by the matrix Σ_D. The model thus does not allow for stochastic volatility or more general changes in the higher moments of D_t over time (though it could handle deterministic changes), nor does it allow for nonlinearities in the time series dependence of D. The fact that we study the level (or change) in fundamentals, rather than their log, is also a restriction, though one that is generally shared by CARA–Normal specifications (e.g. Grossman and Stiglitz (1980)).

There is a set of futures claims on realizations of the fundamental. When we say that the model features a choice of investment across dates or horizons, we mean that investors will choose portfolio allocations across the futures contracts, which then yield exposures to the realization of fundamentals on different dates in the future.

A concrete example of a process D_t is the price of crude oil: oil prices follow some stochastic process and investors trade futures on oil at many maturities. D_t could also be the dividend on a stock, in which case the futures would be claims on dividends on individual dates. The analysis of futures is an abstraction for the sake of the theory, though we note that dividend futures are in fact traded (see Binsbergen and Koijen (2017)). While the concept of a futures market on the fundamentals will be a useful analytic tool, we can also obviously price portfolios of futures. Equity, for example, is a claim to the stream of fundamentals over time. Holding any given combination of futures claims on the fundamental is equivalent to holding futures contracts on equity claims.

1.2 Information structure

There is a unit mass of "sophisticated" or rational investors indexed by $i \in [0,1]$. The realization of the time series of fundamentals, $\{D_t\}_{t=1}^{T}$, can be thought of as a single draw from a multivariate normal distribution. Investors are able to acquire signals about that realization. The signals are a collection $\{Y_{i,t}\}_{t=1}^{T}$ *observed on date 0* with

$$Y_{i,t} = D_t + \varepsilon_{i,t},\ \varepsilon_i \sim N\left(0, \Sigma_i\right), \tag{2}$$

where Σ_i^{-1} is investor i's signal precision matrix (which will be chosen endogenously below). Through $Y_{i,t}$, investors can learn about fundamentals on all dates between 1 and T. $\varepsilon_{i,t}$ is a stationary error process in the sense that $cov\left(\varepsilon_{i,t}, \varepsilon_{i,t+j}\right)$ depends on j but not t. That also implies that $var\left(\varepsilon_{i,t}\right)$ is the same for all t, so all dates are equally difficult to learn about. The stationarity assumption is imposed so that no particular date is given special prominence in the model. Investors must choose an information policy that treats all dates symmetrically, and they are not allowed to choose to learn about a single date.

The signal structure generates one of our desired model features, which is that investors can choose to learn about fundamentals across different dates in the future. When the errors are positively correlated across dates, the signals are relatively less useful for forecasting trends in fundamentals since the errors also have persistent trends. Conversely, when errors are negatively correlated across dates, the signals are less useful for forecasting transitory variation and provide more accurate information about moving averages. What types of fluctuations investors are informed about will determine their investment strategies.

1.3 Investment objective

On date 0, there is a market for forward claims on fundamentals on all dates in the future. Investor i's demand for a date-t forward conditional on the set of prices and signals is denoted $Q_{i,t}$. Investors have mean-variance utility over the net present value of excess returns:

$$U_{0,i} = \max_{\{Q_{i,t}\}} E_{0,i}\left[T^{-1}\beta^t Q_{i,t}\left(D_t - P_t\right)\right] - \frac{1}{2}\left(\rho T\right)^{-1} Var_{0,i}\left[\sum_{t=1}^{T}\beta^t Q_{i,t}\left(D_t - P_t\right)\right], \tag{3}$$

where $0 < \beta \leq 1$ is the discount factor, $E_{0,i}$ and $Var_{0,i}$ are the expectation and variance operators conditional on agent i's date-0 information set, $\{P, Y_i\}$, and ρ is risk-bearing capacity per unit of time. We treat all investors as having identical horizons, T. Appendix B shows that the horizon does not effect information choices in the model. Short- and long-term investors are distinguished by how long they maintain positions, not by their objective – everybody's goal is to earn the highest possible returns, with the least amount of risk, in the shortest time.

The key restriction here (beyond those implicit in the mean-variance assumption) is that signals are acquired and trade occurs on date 0. In general settings there is no known closed-form solution to even the partial-equilibrium dynamic portfolio choice problem, let alone to the full market equilibrium.[9] Moreover, allowing agents to obtain signals repeatedly yields a highly nontrivial updating problem. We therefore use a relatively minimal static model. The model nevertheless has the two characteristics that we stated we desire in the introduction: it allows for investment

[9]Frequency-domain solutions to the infinite regress problem, such as Kasa, Walker, and Whiteman (2013) and Makarov and Rytchkov (2012), restrict preferences to depend on wealth one period ahead in order to avoid the dynamic portfolio problem.

strategies that place different weight on fundamentals on different dates in the future, and it allows investors to make a choice about how precise their signals are for different types of fluctuations in fundamentals.

The time discounting in (3) has the effect of making dates farther in the future less important in the objective of the investors. We therefore define

$$\tilde{Q}_{i,t} \equiv \beta^t Q_{i,t} \tag{4}$$

to be agent i's discounted demand. In what follows, the $\tilde{Q}_{i,t}$ will be stationary processes. That means that $Q_{i,t} = \beta^{-t}\tilde{Q}_{i,t}$ will generally grow in magnitude with maturity t, though only to a relatively small extent for typical values of β and horizons on the order of 10–20 years.

1.4 Noise trader demand

In order to keep prices from being fully revealing, we assume there is uninformed demand from a set of noise traders. The noise traders are investors with the same objective as the sophisticates, but whose expectations are formed differently. Specifically, their expectations of fundamentals depend on an exogenous prior for fundamentals and a signal, which we denote Z_t. The signal Z_t is in reality uncorrelated with fundamentals, so it can be viewed as a type of sentiment shock. The noise traders can potentially also be viewed as uninformed speculators.

Appendix A shows that when the noise traders maximize an objective of the form of (3) but with their incorrect expectations, then their demand, denoted N_t, can be written as

$$\tilde{N}_t = Z_t - kP_t, \tag{5}$$

$$\text{where } \tilde{N}_t \equiv \beta^t N_t. \tag{6}$$

Z_t depends on the signals the noise traders receive (which are assumed to be common across them) and k is a coefficient determining the sensitivity of noise trader demand to prices, which depends on their risk aversion and how precise they believe their signals to be. In principle, N_t can depend on prices on all dates (depending on the structure of priors and signals), but we restrict attention to the case where N_t depends only on P_t for the sake of simplicity.

In the benchmark case where D_t is stationary in levels, we assume that Z_t is also stationary in levels – the noise traders have a signal technology with the same stationarity properties as that of the sophisticates – which yields a useful symmetry between fundamentals, supply, and the signals, in that they are all assumed to be stationary processes. That symmetry is why we use this particular formalization of noise trader demand.

1.5 Asset market equilibrium in the time domain

We begin by solving for the market equilibrium on date 0 that takes the agents' signal precisions, Σ_i^{-1}, as given. The Σ_i^{-1} are chosen on date -1, and that optimization is discussed below.

Definition 1 *For any given set of individual precisions* $\{\Sigma_i\}_{i\in[0,1]}$, *a date-0 asset market equilibrium is a set of demand functions,* $\{Q_i\left(P,Y_i\right)\}_{i\in[0,1]}$, *and a price vector* P, *such that investors maximize utility and all markets clear:* $\int_i Q_{i,t}di + N_t = 0$ *for all* $t \geq 1$.

Investors submit demand curves for each futures contract to a Walrasian auctioneer who selects equilibrium prices to clear all markets. The structure of the time-0 equilibrium is mathematically that of Admati (1985), who studies investment in a cross-section of assets, and the solution from that paper applies directly here (with the minor difference that supply is also a function of prices). Here we are considering investment across a set of futures contracts that represent claims on some fundamentals process across different dates. The Admati (1985) solution is:

$$P = A_1 D + A_2 Z, \tag{7}$$

$$A_1 \equiv I - \left(\rho^2\Sigma_{avg}^{-1}\Sigma_Z^{-1}\Sigma_{avg}^{-1} + \Sigma_{avg}^{-1} + \Sigma_D^{-1} + \rho^{-1}k\right)^{-1}\left(\rho^{-1}k + \Sigma_D^{-1}\right), \tag{8}$$

$$A_2 \equiv \rho^{-1}A_1\Sigma_{avg}^{-1}, \tag{9}$$

$$\text{where } \Sigma_{avg}^{-1} \equiv \int_i \Sigma_i^{-1}di. \tag{10}$$

As Admati (1985) discusses, this equilibrium is not particularly illuminating since standard intuitions, including the idea that increases in demand should raise prices, do not hold. Prices of futures maturing on any particular date depend on fundamentals and demand for all other maturities except in knife-edge cases. Interpreting the equilibrium requires interpreting complicated products of matrix inverses. The following section shows that the equilibrium can be solved nearly exactly by hand when it is rewritten in terms of frequencies.

2 Frequency domain interpretation

2.1 Frequency portfolios

The basic feature of the model that makes it difficult to interpret is that fundamentals, noise trader demand, and signal errors are all correlated across dates. For any one of those three processes, we could always use a standard orthogonal (eigen-) decomposition to yield a set of independent components. But in general there is no reason to expect that three time series with different correlation properties across dates would have the same orthogonal decomposition (in general they do not). A central result from time series analysis, though, is that a particular frequency transform asymptotically orthogonalizes all standard stationary time series processes.

Such a transformation represents simply analyzing the prices of particular portfolios of futures instead of the futures themselves. It must satisfy three requirements. First, the transformation should be full rank, in the sense that the set of portfolios allows an investor to obtain the same payoffs as the futures themselves. Second, the transformed portfolios should be independent of each other. And third, since we are studying trade at different frequencies, it would be nice if the portfolios also had a frequency interpretation.

Obviously there are many different conceptions of fluctuations at different frequencies. One might imagine step functions switching between +1 and -1 at different rates. For reasons we will see below, it turns out that using sines and cosines will be most natural in our setting. So the portfolios that we study – representing investor exposures – vary smoothly over time in the form $\cos(\omega t)$ and $\sin(\omega t)$.

Formally, the portfolio weights are represented as vectors of the form

$$c_h \equiv \sqrt{\frac{2}{T}}\Big(\cos\left(\omega_h\left(t-1\right)\right)\Big)_{t=1}^{T}, \tag{11}$$

$$s_h \equiv \sqrt{\frac{2}{T}}\Big(\sin\left(\omega_h\left(t-1\right)\right)\Big)_{t=1}^{T}, \tag{12}$$

$$\text{where} \quad \omega_h \equiv 2\pi h/T, \tag{13}$$

for different values of the integer $h \in \{0, 1, ..., T/2\}$. c_0 is the lowest frequency portfolio, with the same weight on all dates, while $c_{\frac{T}{2}}$ is the highest frequency, with weights switching each period between $+/-1$.

Figure 1 plots the weights for a pair of those portfolios. The x-axis represents dates and the y-axis is the weight of the portfolio on each date. The weights vary smoothly over time, with the rate at which they change signs depending on the frequency ω.

Economically, the basic idea is to think about the investment problem as being one of choosing exposure to different types of fluctuations in fundamentals. A long-term investor can be thought of as one whose exposure to fundamentals changes little over time. A short-term investor, on the other hand, holds a portfolio whose weights change more frequently and by larger amounts.

Our claim is that studying the frequency portfolios is more natural than studying individual futures claims. Investors do not typically acquire exposure to fundamentals on only a single date. Rather, they have exposures on multiple dates, and the portfolios we study are one way to express that. While investors will also obviously not hold a portfolio that takes the exact form of a cosine, any portfolio can be expressed as a sum of cyclical components. An investor whose portfolio loadings change frequently will have a portfolio whose weights are relatively larger on the high-frequency components, which figure 1 shows generate rapid changes in loadings.

2.2 Properties of the frequency transformation

The portfolio weights can be combined into a matrix, Λ, which implements the frequency transformation.

$$\Lambda \equiv \left[\frac{1}{\sqrt{2}}c_0, c_1, s_1, c_2, s_2, ..., c_{\frac{T}{2}-1}, s_{\frac{T}{2}-1}, \frac{1}{\sqrt{2}}c_{\frac{T}{2}}\right] \tag{14}$$

(s_0 and $s_{T/2}$ do not appear since they are identically equal to zero; the $1/\sqrt{2}$ scaling for c_0 and $c_{T/2}$ gives them the same norms as the other vectors).

We use lower-case letters to denote frequency-domain objects. So whereas $\tilde{Q}_t$ is investor i's vector of discounted allocations to the various futures, $\tilde{q}_i$ is their vector of discounted allocations to the frequency portfolios, with

$$\tilde{Q}_i = \Lambda \tilde{q}_i. \tag{15}$$

Rows of Λ represent portfolio weights on different dates and columns represent different frequency portfolios. $\tilde{q}_i$ is then the vector of investor i's allocations to the various frequency portfolios.

In what follows, we use the index $j = 1, ..., T$ to identify each column of Λ, or equivalently, each frequency-domain vector. The jth column of Λ contains a vector that fluctuates at frequency $\omega_{\lfloor \frac{j}{2} \rfloor} = 2\pi \left\lfloor \frac{j}{2} \right\rfloor / T$, where $\lfloor \cdot \rfloor$ is the integer floor operator.[10] So there are two vectors, a sine and a cosine, for each characteristic frequency, with the exceptions of $j = 1$ (frequency 0, the lowest possible) and $j = T$ (frequency $\frac{T}{2}$, the highest possible).

Note also that Λ has the property that $\Lambda^{-1} = \Lambda'$, so that frequency-domain vectors can be obtained through

$$\tilde{q}_i = \Lambda' \tilde{Q}_i. \tag{16}$$

In the same way that $\tilde{q}_i$ represents weights on frequency-specific portfolios, $d \equiv \Lambda' D$ is a representation of the realization of fundamentals written in terms of frequencies instead of time. The first element of d, for example, is proportional to the realized sample mean of D. Equivalently, d is the set of regression coefficients of D on the columns of Λ (which generate an R^2 of 1).

As a simple example, consider the case with $T = 2$. The low-frequency or long-term component of dividends is then $d_0 = (D_1 + D_2)/\sqrt{2}$ and the high-frequency or transitory component is $d_1 = (D_1 - D_2)/\sqrt{2}$. Agents invest in the low-frequency component d_0 by buying an equal amount of the claims on D_1 and D_2 and they trade the high-frequency component d_1 by buying offsetting amounts of the claims on D_1 and D_2. A short-term investment in this case is one where the sign of the exposure to fundamentals changes, while the long-term investment has a fixed position.

The most important feature of the frequency transformation is that it approximately diagonalizes the variance matrices.

Definition 2 *For an $n \times n$ matrix A with elements $a_{l,m}$, the **weak matrix norm** is*

$$|A| \equiv \left(\frac{1}{n} \sum_{l=1}^{n} \sum_{m=1}^{n} a_{l,m}^2 \right)^{1/2}. \tag{17}$$

If $|A - B|$ is small, then the elements of A and B are close in mean square.

The frequency transform will lead us to study the spectral densities of the various time series:

[10] $\lfloor x \rfloor$ is the largest integer that is less than or equal to x.

Definition 3 *The **spectrum** at frequency ω of a stationary time series X_t is*

$$\begin{aligned} f_X(\omega) &\equiv \sigma_{X,0} + 2\sum_{t=1}^{\infty} \cos(\omega t)\, \sigma_{X,t}, & (18) \\ \text{where } \sigma_{X,t} &= cov(X_s, X_{s-t}). & (19) \end{aligned}$$

The spectrum, or spectral density, is used widely in time series analysis. The usual interpretation is that it represents a variance decomposition. $f_X(\omega)$ measures the part of the variance of X_t associated with fluctuations at frequency ω, which is formalized as follows.

Lemma 1 *For any stationary time series $\{X_t\}_{t=1}^{T}$, with frequency representation $x \equiv \Lambda' X$, the elements of the vector x are approximately uncorrelated in the sense that the covariance matrix of x, $\Sigma_x \equiv \Lambda' \Sigma_X \Lambda$, is nearly diagonal,*

$$|\Sigma_x - diag(f_X)| \leq b T^{-1/2}, \tag{20}$$

and x converges in distribution to

$$x \rightarrow_d N(0, diag(f_X)), \tag{21}$$

where b is a constant that depends on the autocorrelations of X,[11] *and $diag(f_X)$ denotes a matrix with the vector $\left\{f_X\left(\omega_{\lfloor j/2 \rfloor}\right)\right\}_{j=1}^{T}$ on the main diagonal and zeros elsewhere.*[12]

Proof. These are textbook results (e.g. Brockwell and Davis (1991) and Gray (2006)). Appendix C.1 provides a derivation of the inequality (20) specific to our case. The convergence in distribution follows from Brillinger (1981), theorem 4.4.1. ■

Lemma 1 says that Λ approximately diagonalizes *all* stationary covariance matrices. So the frequency-specific components of fundamentals, prices, and noise trader demand are all (approximately) independent when analyzed in terms of frequencies. That is, $d = \Lambda' D$, $y_i = \Lambda' Y_i$, and

[11] Specifically, $b = 4\left(\sum_{j=1}^{\infty} |j\sigma_{X,j}|\right)$.

[12] A requirement of this lemma, which we impose for all the stationary processes studied in the paper, is that the autocovariances are summable in the sense that $\sum_{j=1}^{\infty} |j\sigma_{X,j}|$ is finite (which holds for finite-order stationary ARMA processes, for example). Trigonometric transforms of stationary time series converge in distribution under more general conditions, though. See Shumway and Stoffer (2011), Brillinger (1981), and Shao and Wu (2007).

$z = \Lambda' Z$ all have asymptotically diagonal variance matrices. That independence will substantially simplify our analysis, and it is a special property of the sines and cosines, as opposed to other conceptions of frequencies.[13]

2.3 Market equilibrium in the frequency domain

2.3.1 Approximate diagonalization

Instead of solving jointly for the prices of all futures, the approximate diagonalization result allows us to solve a series of parallel scalar problems, one for each frequency. Intuitively, since the frequency-specific portfolios have returns that are nearly uncorrelated with each other, the investors' utility can be written approximately as a sum of mean-variance optimizations

$$U_{0,i} \approx \max_{\{q_{i,j}\}} T^{-1} \sum_{j=1}^{T} \left\{ E_{0,i}\left[\tilde{q}_{i,j}\left(d_j - p_j\right)\right] - \frac{1}{2}\rho^{-1} Var_{0,i}\left[\tilde{q}_{i,j}\left(d_j - p_j\right)\right] \right\}. \tag{22}$$

In what follows, we solve the model using the approximation for $U_{0,i}$, and then show that it converges to the true solution from Admati (1985). When utility is completely separable across frequencies, there is an equilibrium frequency by frequency:

Solution 1 *Under the approximations $d \sim N\left(0, diag\left(f_D\right)\right)$ and $z \sim N\left(0, diag\left(f_Z\right)\right)$, the prices of the frequency-specific portfolios, p_j, satisfy, for all j*

$$p_j = a_{1,j} d_j + a_{2,j} z_j \tag{23}$$

$$a_{1,j} \equiv 1 - \frac{\rho^{-1} k + f_{D,j}^{-1}}{\left(\rho f_{avg,j}^{-1}\right)^2 f_{Z,j}^{-1} + f_{avg,j}^{-1} + f_{D,j}^{-1} + \rho^{-1} k} \tag{24}$$

$$a_{2,j} \equiv \frac{a_{1,j}}{\rho f_{avg,j}^{-1}} \tag{25}$$

where $f_{avg,j}^{-1} \equiv \int_i f_{i,j}^{-1} di$ is the average precision of the agents' signals at frequency j.

Proof. See appendix C.2. ■

[13] Finally, it is should be noted that infill asymptotics, where T grows by making the length of a time period shorter, are not sufficient for lemma 1 to hold. What is important is essentially that T is large relative to the range of autocorrelation of the process X. So, for example, if fundamentals have nontrivial autocorrelations over a horizon of a year, then it is important that T be substantially larger than a year. Van Binsbergen and Koijen (2017), for example, examine data on dividend futures with maturities as long as 16 years.

The price of the frequency-j portfolio depends only on fundamentals and supply at that frequency due to the independence across frequencies. As usual, the informativeness of prices, through $a_{1,j}$, is increasing in the precision of the signals that investors obtain, while the impact of noise trader demand on prices is decreasing in signal precision and risk tolerance.

These solutions for the prices are the standard results for scalar markets. What is different here is simply that the agents chose exposures across frequencies, rather than across dates; p_j is the price of a portfolio whose exposure to fundamentals fluctuates over time at frequency $2\pi \lfloor j/2 \rfloor /T$. Both prices and demands at frequency j depend only on signals and supply at frequency j – the problem is completely separable across frequencies.

In what follows, we assume that k is sufficiently small that $ka_{2,j} < 1$ for all j, which simply ensures that when z represents a positive demand shock in equilibrium (though most of the results hold without that assumption). The restriction is that noise trader demand not be too sensitive to prices; in the literature k is usually equal to zero.

2.3.2 Quality of the approximation

While solution 1 is an approximation, its error can be bounded. The time domain solution is obtained from the frequency domain solution by premultiplying by Λ (from equation (15)), and we have,

Proposition 1 *The difference between solution 1 and the exact Admati (1985) solution is small in the sense that*

$$\left|A_1 - \Lambda diag\left(a_1\right)\Lambda'\right| \leq c_1 T^{-1/2} \tag{26}$$

$$\left|A_2 - \Lambda diag\left(a_2\right)\Lambda'\right| \leq c_2 T^{-1/2} \tag{27}$$

for constants c_1 and c_2. Furthermore, the variances of the approximation error for prices and quantities are bounded by:

$$\left|Var\left(\Lambda p - P\right)\right| \leq c_P T^{-1/2} \tag{28}$$

$$\left|Var\left(\Lambda \tilde{q}_i - \tilde{Q}_i\right)\right| \leq c_Q T^{-1/2} \tag{29}$$

for constants c_P and c_Q.

Proof. See appendix C.3. ■

Proposition 1 shows that the frequency domain solution to the market equilibrium provides a close approximation to the true solution in the sense that the solution in (23), once it is rotated back to the time domain, converges to equations (7–9). Moreover, Λp is stochastically close to P in the sense that the variance of the pricing errors is of order $T^{-1/2}$. So the standard time-domain solution for stationary time series processes becomes arbitrarily close to a simple set of parallel scalar problems in the frequency domain for large T.

2.4 Optimal information choice in the frequency domain

The analysis so far takes the precision of the signals as fixed. Following Van Nieuwerburgh and Veldkamp (2009) and Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016), we allow investors to choose their signal precisions, Σ_i^{-1} to maximize the expectation of their mean-variance objective (3) subject to an information cost,

$$\max_{\{f_{i,j}\}} E_{-1}\left[U_{i,0} \mid \Sigma_i^{-1}\right] - \frac{\psi}{2T} tr\left(\Sigma_i^{-1}\right), \tag{30}$$

where E_{-1} is the expectation operator on date -1, i.e. prior to the realization of signals and prices (as distinguished from $E_{i,0}$, which conditions on P and Y_i), and ψ is the per-period cost of information. Total information here is measured by the trace operator $tr\left(\Sigma_i^{-1}\right)$.[14]

Given the optimal demands, an agent's expected utility is linear in the precision they obtain at each frequency.

Lemma 2 *Each informed investor's expected utility at time -1 may be written as a function of their own signal precisions, $f_{i,j}^{-1}$, and the average across other investors, $f_{avg,j}^{-1} \equiv \int_i f_{i,j}^{-1} di$, with*

$$E_{-1}\left[U_{0,i} \mid \{f_{i,j}\}\right] = \frac{1}{2T} \sum_{j=1}^{T} \lambda_j \left(f_{avg,j}^{-1}\right) f_{i,j}^{-1} + constant, \tag{31}$$

[14] Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016) show that the results here are robust to various perturbations of the assumptions: (1) rather than using the trace operator, information can be measured through the entropy of the signals; (2) investors can be given a fixed budget of information rather than a fixed cost; (3) it can be made costly for investors to pay attention to prices in addition to their signals.

where the constant does not depend on investor i's precision.

Proof. See appendix C.4. ■

$\lambda_j(x)$ is a function determining the marginal benefit of information at each frequency, with the properties $\lambda_j(x) > 0$ and $\lambda_j'(x) < 0$ for all $x \geq 0$. It is possible to show that $\lambda_j\left(f_{avg,j}^{-1}\right) = Var\left[d_j - p_j\right]$.[15]

Since expected utility and the information cost are both linear in the set of precisions that agent i chooses, $\left\{f_{i,j}^{-1}\right\}$, it immediately follows that agents purchase signals at whatever subset of frequencies has $\lambda_j\left(f_{avg,j}^{-1}\right) \geq \psi$.

Solution 2 *Information is allocated so that*

$$f_{avg,j}^{-1} = \begin{cases} \lambda_j^{-1}(\psi) & \textit{if } \lambda_j(0) \geq \psi, \\ 0 & \textit{otherwise.} \end{cases} \tag{32}$$

Because attention cannot be negative, when $\lambda_j(0) \leq \psi$, no attention is allocated to frequency j. Otherwise, attention is allocated so that its marginal benefit and its marginal cost are equated. Note, though, that this result does not pin down precisely how any specific investor's attention is allocated; this class of models, with a non-convex information cost, only determines the aggregate allocation of attention across frequencies. For the purposes of studying price informativeness, though, characterizing this aggregate allocation is all that is necessary.

Solution 2 is the water-filling equilibrium of Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016). In their case it applied to the variances of principal components of a cross-section of assets, where here it applies to variances of frequency portfolios – the spectrum.

At this point there are still no investors who are explicitly "short-term" or "long-term". Investors can follow many different strategies, with different mixes of short- and long-term focus. Even without any specialization to particular strategies, though, we now have sufficient structure to analyze the effects of restrictions on the strategies that investors may follow. Later on, we explicitly discuss how to think about short- and long-term investors.

[15] These results are established as part of the proof of lemma 2, in appendix C.4.

3 The consequences of restricting investment frequencies for prices

This section focuses on the effects on prices of restrictions on the frequencies at which investment strategies can operate. It examines a particularly stark restriction that simply outlaws certain strategies. Section 4 studies information restrictions, while section 5 shows that the results here are similar to those obtained by imposing a quadratic tax on trading.

3.1 Restricting investment frequencies

The assumption in this section is that investors are restricted to setting $\tilde{q}_{i,j} = 0$ for j in some set $\mathcal{R}$. We leave the noise traders unconstrained, assuming they are perhaps less regulated (like retail investors, in many ways), or that their demand is induced by a rotating set of people, so that variation in aggregate noise trader demand does not correspond necessarily to variation in the demand of any individual.

Intuitively, if an investor is restricted from exposures at frequencies shorter than a day (i.e. $\mathcal{R}$ is the set of frequencies corresponding to cycles lasting less than one day), then they can effectively only choose exposures once per day. Rather than forcing the investor to literally only trade once a day, though, the restriction in our case corresponds to a portfolio that varies smoothly between days. So (approximately) if the investor can choose daily exposures, then their actual exposures, minute-by-minute, might be represented by a spline that smooths between the daily exposures.

More formally, a restriction on exposures to the frequency portfolios reduces the degrees of freedom that an investor has in making choices. Suppose we had a model where each time period is an hour, and T is a year, or 1625 trading hours. A restriction that investors cannot invest at a frequency higher than a day (6.5 hours) would mean that they would go from a strategy with 1625 degrees of freedom to one with only 250. A pension that sets a portfolio once a quarter would have only four degrees of freedom. In that sense, then, the restrictions we analyze in this section are similar to a shift from a continuous market to one with infrequent batch auctions, as in Budish, Cramton, and Shim (2015). While that paper proposes holding the auctions still very frequently (i.e. more than once per second), a more aggressive restriction could have auctions only once per day, or once per hour.

Appendix G examines the version of the model in which fundamentals are stationary in differences instead of levels (i.e. they have a unit root). In that case, the analysis goes through nearly identically – frequency restrictions still represent decreases in the degrees of freedom available to investors – but with a single small change: the lowest frequency portfolio, rather than being one that puts equal weight on fundamentals on all dates, puts weight on fundamentals only on the final date, T. Intuitively, an investor who wants to take a position in long-run growth rates does that by buying a claim just to the level on date T. On the other hand, an investor who holds a portfolio that loads on rapid changes in the growth rate of fundamentals will have a portfolio with weights on the level of fundamentals that also change quickly. So in that case, the example of restricting investment in portfolios with frequencies higher than a day continues to impose the same limit on the set of strategies investors can choose from.

Derivations of the results in the remainder of this section are in appendix D.

3.2 Results

We begin by describing price informativeness at different frequencies to demonstrate our key separation result. We then show what happens to prices of standard claims in the time domain.

3.2.1 Price informativeness across frequencies

In terms of frequencies, there is a complete separation: prices become uninformative at restricted frequencies, while remaining unaffected at unrestricted frequencies.

Result 1 *When investment by sophisticated investors is restricted at a set of frequencies $\mathcal{R}$, prices satisfy*

$$p_j = \left\{ \begin{array}{c} k^{-1} z_j \text{ for } j \in \mathcal{R} \\ a_{1,j} d_j + a_{2,j} z_j \text{ otherwise} \end{array} \right\}, \tag{33}$$

where a_1 and a_2 are the same as those defined in solution 1.

Intuitively, when sophisticated investors are restricted, prices depend only on sentiment, since the people with information cannot express their opinions. Moreover, the market becomes illiquid, and it is cleared purely through prices rather than quantities.

Since the solution for information acquisition at a frequency j does not depend on anything about any other frequency, the information acquired at a frequency $j \notin \mathcal{R}$ is also unaffected by the policy. We then have the result that:

Corollary 1.1 *When investors are restricted from holding portfolios with weights that fluctuate at some set of frequencies $j \in \mathcal{R}$, then prices at those frequencies, p_j, become completely uninformative about dividends. The informativeness of prices for $j \notin \mathcal{R}$ about dividends is unchanged. More formally, $Var[d_j \mid p_j]$ for $j \notin \mathcal{R}$ is unaffected by the restriction. For $j \in \mathcal{R}$, $Var[d_j \mid p_j] = Var[d_j]$.*

So a policy that eliminates short-term investment, e.g. by requiring holding periods of some minimum length, reduces the informativeness of prices for the short-term or transitory components of fundamentals, but has no effect on price informativeness in the long-run.[16]

3.2.2 Price informativeness across dates

The fact that prices remain equally informative at some frequencies does not mean that they remain equally informative for any particular *date*. Dates and frequencies are linked through

$$Var(D_t \mid P) = \frac{1}{T}\sum_{j=1}^{T} Var[d_j \mid p_j]. \tag{34}$$

The variance of an estimate of fundamentals conditional on prices at a particular date is equal to the average of the variances across all frequencies.[17] So when uncertainty rises at some set of frequencies, the informativeness of prices for fundamentals on every date falls by an equal amount.

Corollary 1.2 *Investment restrictions reduce price informativeness for fundamentals on all dates by equal amounts, and by an amount that weakly increases with the number of frequencies that are restricted.*

[16]This result is very slightly sensitive to the specification of the information constraint. The model assumes investors face a fixed cost of information acquisition. Alternatively, Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016) assume that investors have a fixed budget of information. In that case, when investment is restricted at a particular frequency, information is reallocated to the unrestricted frequencies, with the consequence that informativeness still falls to zero at restricted frequencies, but it actually rises at the unrestricted frequencies, instead of remaining unchanged.

[17]This result is proven in appendix 4.2.

If a person is making decisions based on estimates of fundamentals from prices and they are worried that prices are contaminated by high-frequency noise due to a restriction on short-term investment, a natural response would be to examine an average of fundamentals and prices over time (across maturities of futures contracts).

Corollary 1.3 *The informativeness of prices for the sum of fundamentals depends only on informativeness at the lowest frequency:*

$$Var\left(T^{-1}\sum_{t=1}^{T} D_t \mid P\right) = Var\left[T^{-1/2}d_0 \mid p_0\right], \tag{35}$$

where d_0 is the lowest frequency portfolio – with equal weight each date – and p_0 is its price.

Result 1.3 follows immediately from the definition of d_0 and the independence across frequencies in the solution. It shows that the informativeness of prices for moving averages of fundamentals depends only on the very lowest frequency. So even if prices have little or no information at high frequencies – $Var\left[d_j \mid p_j\right]$ is high for large j – there need not be any degradation of information about averages of fundamentals over multiple periods, as they depend primarily on precision at lower frequencies (smaller values of j).

More concretely, going back to our example of oil futures, when investors are not allowed to choose exposure to the high-frequency portfolios, prices become noisier, making it more difficult to obtain an accurate forecast of the spot price of oil at some specific moment in the future. But if one is interested in the average of spot oil prices over a year, on the other hand, then we would expect futures prices to remain informative under restrictions on short-term strategies. It is possible to derive a similar result for moving averages shorter than T; in that case the weights on the frequencies are given by the Fejér kernel.

In the case where fundamentals are stationary in terms of growth rates instead of levels, the results in this section also hold, but replacing D_t by its first difference. In particular, result 1.3 then states that $Var(D_T \mid P)$ is equal to the variance of the lowest frequency portfolio. This is unsurprising since, as we had previously noted, in the difference-stationary case, the lowest frequency portfolio is the one that places weight only on D_T. In that case, the prediction of the model is that $Var\left(D_T \mid P\right)$ is unaffected by restrictions on short-term investment.

When long-run investment strategies are restricted, on the other hand, as in the case of a trading desk that cannot have exposure to cycles lasting longer than a day (e.g. Brock and Kleidon (1992) and Menkveld (2013)), then it is natural to examine the informativeness of *differences* in prices across dates. As an example, we can consider the variance of the first difference of fundamentals.

Corollary 1.4 *The variance of an estimate of the change in fundamentals across dates conditional on observing the vector of prices is*

$$Var\left[D_t - D_{t-1} \mid P\right] = \sum_{j=1}^{T} 2\left(1 - \cos\left(\omega_{\lfloor j/2 \rfloor}\right)\right) Var\left[d_j \mid p_j\right]. \tag{36}$$

The function $2 - 2\cos(\omega)$ is equal to 0 at $\omega = 0$ and rises smoothly to 4 at the highest frequency, $\omega = \pi$. So period-by-period changes in fundamentals are driven primarily by high-frequency variation. Reductions in price informativeness at low frequencies then have relatively large effects on moving averages and small effects on changes, while the reverse is true for reductions in informativeness at high frequencies.

To summarize, any restriction on investment reduces price informativeness for any particular date. But when short-term investment is restricted, there is little change in the behavior of moving averages of prices. So if a manager is making investment decisions based on fundamentals only at a particular moment, then that decision will be hindered by the policy since prices now have more noise. But if decisions are made based on averages of fundamentals over longer periods, the model predicts that there need not be adverse consequences.

3.2.3 Return volatility

Corollary 1.5 *Given an information policy $f_{avg,j}^{-1}$, the variance of returns at frequency j, $r_j \equiv d_j - p_j$ is*

$$Var\left(r_j\right) = \begin{cases} f_{D,j} + \frac{f_{Z,j}}{k^2} \text{ for } j \in \mathcal{R} \\ \min\left(\psi, \lambda_j\left(0\right)\right) \text{ otherwise} \end{cases}. \tag{37}$$

Moreover, the variance of returns at restricted frequencies satisfies $Var(r_j) > f_{D,j} + \frac{f_{Z,j}}{(k+\rho f_{D,j}^{-1})^2}$, which is the variance that returns would have at the same frequency if investment were unrestricted but agents were uninformed.

The volatility of returns at a restricted frequency is higher than it would be if the sophisticated investors were allowed to trade, even if they gathered no information. Intuitively, when uninformed active investors have risk-bearing capacity ($\rho > 0$), they absorb some of the exogenous demand by simply trading against prices, buying when prices are below their means and selling when they are above. The greater is the risk-bearing capacity, the smaller is the effect of sentiment volatility on return volatility. Thus, the restriction affects return volatilities through its effects on both liquidity provision and information acquisition.

Restricting sophisticated investors from following short-term strategies in this model can thus substantially raise asset return volatility in the short-run – it can lead to, for example, large day-to-day fluctuations in prices (though those fluctuations in prices are, literally, variations in prices across maturities for different futures contracts on date 0). Sophisticated traders typically play a role of smoothing prices across maturities, essentially intermediating between excess demand on one day and excess supply in the next. When they are restricted from holding positions in futures that fluctuate from day to day, they can no longer provide that intermediation service, and short-term volatility increases. So while there might be other reasons why one might want to restrict short-term investment (e.g. due to incentive effects on managers, as in Shleifer and Vishny (1990), or reducing losses of noise traders, as we discuss below), a consequence will be that transitory and inefficient price volatility will increase.

Finally, we note that the results in this section could be extended fairly easily to account for more general types of restrictions, such as placing restrictions only on the trade of a subset of agents, or perhaps bounding the size of the positions of some agents at certain frequencies.[18]

4 Investor outcomes

This section studies how restricting short-term investment affects long-term investors. We obtain two main results for the sophisticated investors, which initially appear to be in conflict:

1. The entrance of short-term investors reduces the profits of long-term investors.
2. Restricting short-term investment reduces the profits and utility of long-term investors.

[18] See Dávila and Parlatore (2018) for an extensive analysis of the relationship between informativeness and volatility.

So while long-term investors are worse off when short-term investors enter the market, cutting off short-term investment strategies (the ability to rapidly turn over portfolios) neither restores the old equilibrium, nor does it make the long-term investors better off.

We also examine the implications of the policies for noise traders, finding that noise traders are generally best off when prices are most informative.

4.1 Who are short- and long-term sophisticated investors?

We define a short-term investor as one whose portfolio is driven relatively more by high-frequency fluctuations, while a long-term investor holds a portfolio that is driven relatively more by low-frequency fluctuations. That definition can be formalized by a variance decomposition, using the facts

$$Var\left(\tilde{Q}_{i,t}\right) = \sum_{j=1}^{T} Var\left(\tilde{q}_{i,j}\right) \tag{38}$$

$$\frac{d}{df_{i,j}^{-1}}\left[Var\left(\tilde{q}_{i,j}\right)\right] > 0 \tag{39}$$

(where the second line is derived by a direct calculation from the demand function derived in section C.2.2). The component of the variance of $\tilde{Q}_{i,t}$ that is driven by fluctuations at frequency j, $Var\left(\tilde{q}_{i,j}\right)$, is increasing in the precision of the signals agent i acquires at frequency j ($f_{i,j}^{-1}$). So if two investors have the same total variance of their positions, $Var\left(\tilde{Q}_{1,t}\right) = Var\left(\tilde{Q}_{2,t}\right)$, but one of them has higher-precision signals at high frequencies, i.e. $f_{1,j}^{-1} > f_{2,j}^{-1}$ for j above some cutoff, then variation in that investor's position is driven relatively more by high-frequency components.

(39) shows that $Var\left(\tilde{q}_{i,j}\right)$ is increasing in the precision of the signals that agent i receives. When an investor has more precise signals at a given frequency, they trade more aggressively for two reasons. First, since their signals are more precise, their demand is more sensitive to their own signals. Second, the quality of their signals also means that they can worry less about adverse selection, so they trade more strongly to accommodate demand shocks from noise traders.

For two investors with positions that have the same unconditional variance, the short–term investor – whose fluctuations happen relatively faster – is the one with relatively more precise signals about the transitory or high-frequency features of fundamentals. That is, *short-term investors* *have short-term/high-frequency information, and long-term investors have long-term/low-frequency information.* As an extreme case, we will take short-term investors as people whose signals have positive precision only for j above some cutoff j_{HF}, and long-term investors have signals with positive precision only for j below some j_{LF} with $j_{HF} > j_{LF}$.

4.2 Investor profits and utility

Result 2 *Let $R = D - P$ be the vector of returns in the time domain. Investor i's average discounted profits are*

$$E_{-1}\left[\tilde{Q}_i' R\right] = \sum_{j=1}^{T} (1 - ka_2)\left(-E_{-1}\left[z_j r_j\right]\right) + ka_1 E_{-1}\left[r_j d_j\right] + \rho\left(f_{i,j}^{-1} - f_{avg,j}^{-1}\right) Var_{-1}\left[r_j\right] \tag{40}$$

and expected profits at each frequency are nonnegative,

$$E_{-1}\left[\tilde{q}_{i,j} r_j\right] \geq 0 \text{ for all } i, j \tag{41}$$

with equality only if $f_{i,j} = 0$ and $f_{D,j}^{-1} = \rho f_{avg,j}^{-1} f_{Z,j}^{-1} k$ (i.e. in a knife-edge case).

Each investor's expected discounted profits depend on three terms. The first represents the profits earned from noise traders. $E\left[z_j r_j\right] = -a_2 f_Z^{-1} < 0$ since the sophisticated investors imperfectly accommodate their demand. When the noise traders have high demand (that is, when z is high), they drive prices up and expected returns down. The sophisticated investors earn profits from trading with that demand.

The second term represents the profits that the informed investors earn by buying from the noise traders when they have positive signals on average. The coefficient $ka_{1,j}$ represents the slope of the supply curve that the informed investors face.

Finally, the third term in (40) represents a reallocation of profits from the less to the more informed sophisticated investors. An investor who has highly precise signals about fundamentals at frequency j can accurately distinguish periods when prices are high due to strong fundamentals from those when prices are high due to high sentiment. That allows them to earn relatively more profits on average than an uninformed investor.

That said, an uninformed sophisticated investor does not earn negative expected profits at any frequency, even with $f_{i,j}^{-1} = 0$. There are always, except in a knife-edge case, profits to be earned by trading with noise traders.

Result 2 therefore gives us two key insights. First, all investors, no matter their information, have the ability to earn profits at all frequencies through liquidity provision. Second, all else equal, investors who are informed about a particular frequency earn the most money from investing at these frequencies. Short-term investors – those with relatively more information about high-frequency fundamentals – earn relatively higher returns at high frequencies, while long-term investors earn relatively higher returns at low frequencies.

4.2.1 The entrance of short-term investors

The two main results of this section follow from result 2. First, consider a scenario in which $f_{avg,j}^{-1} = 0$ for high frequencies (i.e. for all j greater than j_{HF}). That is, there are initially no short-term investors, perhaps because an unmodeled cost of acquiring high-frequency information is prohibitively large. Existing investors may trade at those frequencies $j > j_{HF}$, but they do so without any information. What is the effect of the initial entry of short-term investors, i.e. a marginal increase in $f_{avg,j}^{-1}$ for $j > j_{HF}$, holding all other parameters fixed?

Corollary 2.1 *Starting from $f_{avg,j}^{-1} = 0$ for $j > j_{HF}$, an increase in $f_{avg,j}^{-1}$ at one of those frequencies reduces profits and utility of an investor for whom $f_{i,j}^{-1}$ remains unchanged. Specifically,*

$$\left.\frac{d}{df_{avg,j}^{-1}} E_{-1}\left[\tilde{q}_{LF,j} r_j\right]\right|_{f_{avg,j}=0} < 0 \tag{42}$$

$$\left.\frac{d}{df_{avg,j}^{-1}} E_{-1}\left[\sum_t \tilde{Q}_{LF,t}\left(D_t - P_t\right)\right]\right|_{f_{avg,j}=0} < 0 \tag{43}$$

$$\left.\frac{d}{df_{avg,j}^{-1}} E_{-1}\left[U_{LF,0}\right]\right|_{f_{avg,j}=0} < 0 \tag{44}$$

where the LF subscripts denote positions and utility of a long-term investor. Concretely, in an economy populated only by long-term investors, the entrance of short-term investors increases $f_{avg,j}^{-1}$ for $j > j_{HF}$ and therefore reduces the expected profits at those frequencies, total expected profits, *and the utility of long-term investors.*

The source of that result is the fact that investors with low-frequency information may still invest in the short-run (i.e. have exposures that change from day to day). Suppose, for example, that not only does $f_{LF,j}^{-1} = 0$ for high j, but also that $f_{avg,j}^{-1}$ does also – nobody has short-term information. In that setting obviously any sophisticated investor will be happy to accommodate transitory fluctuations in noise trader demand. More concretely, an investor who has information that the long-term value of a stock is $50 will be willing to provide liquidity in the short-run, buying when the price is below $50 and selling when the price is higher. That liquidity provision will have high-frequency components when liquidity demand (noise trader demand) has high-frequency components (i.e. $f_{z,j} > 0$ for $j > j_{HF}$). That is, if there are short-run variations in sentiment, then there will be short-run variation in the low-frequency investor's position.

The entry of investors with high-frequency information hurts those with low-frequency information because the new investors are better at providing short-term liquidity. Result 2 and corollary 2.1 formalize that idea and shows that how short-term investors hurt long-term investors – by crowding out their ability to provide liquidity. It is critical to note, though, that result 2 still shows that the entry of short-term investors never reduces the profits earned by long-term investors to zero, even at high frequencies.

It should also be noted that these results do not change the incentives of low-frequency investors to acquire information at low frequencies. While they lose money from a decrease in liquidity provision at high frequencies, their choices at low frequencies are unaffected, so if one's primary concern is price informativeness at low frequencies, the entry of short-term investors will have no effect.

Finally, we also note that the entrance of short-term investors has positive effects on the overall market:

Corollary 2.2 *The entry of short-term investors, increasing* $f_{avg,j}^{-1}$ *for* $j > j_{HF}$ *increases price* *informativeness and reduces return volatility at those frequencies. That is, for any frequency*

$$\frac{d}{df_{avg,j}^{-1}} Var\left[d_j \mid p_j\right] \leq 0 \tag{45}$$

$$\frac{d}{df_{avg,j}^{-1}} Var\left[r_j - p_j\right] \leq 0 \tag{46}$$

So while long-term investors may be hurt by the entry of the short-term investors, to a regulator whose goal is simply to maximize price informativeness or minimize return volatility, the short-term investors are beneficial.

4.2.2 Policy responses

If the entrance of short-term investors hurts the incumbent long-term investors, a natural question to the incumbents might be how to restore the old equilibrium. We consider three responses that have been proposed: restricting or eliminating short-term investment, taxing transactions (or variation in positions), and limiting the availability of short-term information.

First, consider a restriction on short-term/high-frequency investment that says that no sophisticated investor may set $q_{i,j} \neq 0$ for j above some cutoff, as in the previous section. A concrete example of such a policy is an infrequent batch auction mechanism, similar to Budish, Cramton, and Shim (2015). Restricting investment above the daily frequency would correspond to having an auction once per day. Result 2 shows that such a restriction would, rather than restoring the profits and utility of the long-term investors, actually reduce them *further*. The result follows from the fact that restricting investment eliminates the terms in the summation for j above the cutoff, which are all nonnegative. While short-term investors make liquidity provision at high frequencies more difficult, outlawing short-term investment simply makes it impossible. So eliminating short-term investment does not restore the old equilibrium – it actually compounds the effect of the entrance of short-term investors.

Corollary 2.3 *Limiting short-term investment with a policy that restricts sophisticated investors from holding $q_{i,j} \neq 0$ for $j > j_{HF}$ weakly reduces the profits and expected utility of all investors.*

Imposing a tax on changes in positions, specifically, a tax on $(Q_{i,t} - Q_{i,t-1})^2$, will have similar effects to a restriction on short-term investment in that the tax is most costly for short-term strategies with high turnover. The next section provides a more complete derivation of that result.

The final policy response would be to somehow limit the acquisition of high-frequency information. In the context of the model, that would represent a restriction on the ability of investors to learn about period-to-period variation in fundamentals, for example by making it more costly to acquire high-frequency information. A simple generalization of the model is to assume that instead of the cost of information at each frequency being ψ, it is instead indexed by frequency, ψ_j. In that case, information is allocated according to the rule

$$f_{avg,j}^{-1} = \begin{cases} \lambda_j^{-1}(\psi_j) & \text{if } \lambda_j(0) \geq \psi_j, \\ 0 & \text{otherwise.} \end{cases} \tag{47}$$

and we have from the properties of λ that

$$\frac{df_{avg,j}^{-1}}{d\psi_j} \leq 0 \tag{48}$$

So an increase in ψ_j has the direct effect of reducing overall information acquisition at frequency j.

A specific example of a policy that makes it more costly for investors to acquire high-frequency information could come from a reduction in the information that firms freely release. For example, there have been suggestions to change financial reporting requirements so that less short-run information is revealed proposed by the CFA institute (Schacht et al. (2007)) and Brookings Institution (Pozen (2014)). In the UK quarterly earnings reports are no longer mandatory, and Gigler et al. (2014) argue that reducing reporting frequency can reduce managerial biases toward short-term projects.

When firms stop reporting quarterly earnings, or providing short-term earnings guidance, they are at the very least making information acquisition more expensive – instead of simply reading announcements, investors now must do research to try to measure short-term performance. And it is possible that some of the information is simply no longer even able to be acquired by investors (i.e. inside information), which might correspond to $\psi_j \to \infty$.

In the context of the model, a restriction on information acquisition could in fact exactly restore the equilibrium that exists in the absence of the short-term investors. Since the long-term investors do not acquire high-frequency information, the restriction has no direct effect on them. In terms of the results above, the reason that short-term investors harm long-term investors in the model is that they increase $f_{avg,j}^{-1}$ for high values of j. A policy that makes short-term information more expensive does the opposite, reducing $f_{avg,j}^{-1}$ and shifting the market back to the previous equilibrium.

To be clear, the claim here is not that information should be restricted or that markets should be tilted in the direction of long-term investors (and given the results above on price informativeness and return volatility, those restrictions seem counterproductive). Rather, the goal is simply to understand the effects of such policies, since they have been proposed and discussed widely.

4.3 Outcomes for noise traders

The formalization of noise traders used here is that they are investors whose demand depends on an uninformative signal that they erroneously believe forecasts fundamentals. Under that interpretation, a natural objective of a policymaker might be to set policies to try to reduce the losses of these investors. That is, if one thinks that retail investors trade based on sentiment, then one might want to try to limit their losses. Relatedly, the noise traders might be interpreted as representing uninformed speculators. The goal then would be to reduce their losses and try to keep speculative demand from affecting prices and creating volatility (that is the motivation of the transaction tax in Tobin (1978)).

The policies examined in the previous section have direct implications for the losses of noise traders. First, note that the average returns that noise traders earn must be exactly the opposite of what the informed investors earn on average (i.e. equation (40) with $f_{i,j}^{-1} = f_{avg,j}^{-1}$):

Corollary 2.4 *The average earnings of noise traders are*

$$E\left[\sum_{t=1}^{\infty} \tilde{N}_t R_t\right] = \sum_j \underbrace{\left(\left(1 - ka_{2,j}\right) z_j - ka_1 d_j\right)}_{\textit{Noise trader demand}} \underbrace{\left(\left(1 - a_{1,j}\right) d_j - a_{2,j} z_j\right)}_{\textit{Returns}} \tag{49}$$

$$= -\sum_j \left[a_{2,j} \left(1 - ka_{2,j}\right) f_{Z,j} + ka_{1,j} \left(1 - a_{1,j}\right) f_{D,j}\right] \tag{50}$$

Average noise trader earnings are quadratic in the coefficients determining prices, $a_{1,j}$ and $a_{2,j}$. That is caused by the interaction of two effects. First, when expected returns are more responsive to their demand shocks ($a_{2,j}$ is large) or to fundamentals ($1 - a_{1,j}$ is large), then expected returns vary more, giving more potential for losses. However, variation in prices inhibits their trading since they have downward sloping demand curves, with slope k. So when $1 - ka_{2,j}$ is small or $ka_{1,j}$ is small, losses are smaller.

There are thus two ways to drive the losses of noise traders to zero. One is for prices to be optimal, or completely informative, with $a_{1,j} = 1$ and $a_{2,j} = 0$ (i.e. $p_j = d_j$). That case is obviously ideal in that noise traders have no losses and prices are also most useful as signals for making decisions. Noise trader losses are zero in this case because informed investors have perfectly elastic demand curves, and will trade any quantity since they know the price is exactly equal to fundamentals. The second way to reduce noise trader losses to zero is to drive $a_{1,j}$ to zero and $a_{2,j} = k^{-1}$. In that case, prices are completely uninformative, and they move in such a way that there is no trade. This achieves the goal of minimizing noise trader losses, but at the cost of eliminating all information from asset markets.

4.3.1 Restricting investment

The two policies examined above – restricting trade and restricting information – both drive in the direction of the second way to reduce noise trader losses. Restricting all investment by the informed investors at a given frequency eliminates all information from prices, but it also means that the noise traders have nobody to trade with, so their losses are identically zero. Similarly, restricting information, by reducing $f_{avg,j}^{-1}$ to zero, sets $a_{1,j}$ to zero, so that the noise traders have no losses due to the information held by the sophisticated investors (the second part of equation (50)). We also have

$$f_{avg,j} = 0 \Rightarrow 1 - ka_{2,j} = \frac{f_D^{-1}}{\rho^{-1}k + f_D^{-1}} \tag{51}$$

The noise traders will still lose money to the informed investors in general, through the first term in equation (50). As the amount of fundamental uncertainty grows, though – f_D^{-1} shrinks – the losses eventually fall to zero.

So unlike above, for the purpose of protecting noise traders, instead of long-horizon investors, the trading restriction is more effective than the information restriction. The information restriction does not in general reduce the losses of the noise traders to zero, while the trade restriction does.

Either policy is only second-best, though, in the sense that they help noise traders by reducing the informativeness of prices and increasing price volatility.

The policy of restricting investment would be most natural if there were some frequencies at which f_Z was particularly large and f_D particularly small. At such a frequency, the information loss from restricting investment is relatively small – in fact it could potentially even be zero if $\lambda_j(0)$ is sufficiently small (since there would also be no information acquisition in the absence of the restriction) – and the benefit is relatively large, since it increases in f_Z (equation (50)). So restrictions on investment make the most sense at frequencies with little variation in fundamentals but substantial variation in sentiment or noise trader demand.

Corollary 2.5 *At any frequency where $f_{Z,j}$ is sufficiently large or $f_{D,j}$ sufficiently small that $\lambda_j(0) \leq \psi$ (recall that $\lambda_j(0)$ represents the marginal value of acquiring information when $f_{avg,j}^{-1} = 0$)[19], there is no information acquisition in equilibrium and prices are completely uninformative. At those frequencies, restricting trade by mandating that $q_{i,j} = 0$ reduces the losses of noise traders to zero and has no effect on price efficiency, since prices are already uninformative.*[20]

A common view is that there is relatively little important economic news at high frequencies since economic decisions such as physical investment depend on relatively long-term expectations. In such a case, one would think that $f_{D,j}$ is small at high frequencies. The results here then show that it would be natural to restrict high-frequency investment since there is no information loss and the effects of noise trader demand or speculation are eliminated. The model here formalizes that common intuition.

4.3.2 Subsidizing information

On the other hand, if the goal was to reduce noise trader losses without any reduction in price informativeness, then the ideal policy would be one that *increases* $f_{avg,j}^{-1}$. Specifically, as the quantity

[19] Specifically, $f_{Z,j} < \left(\psi - f_{D,j}^{-1}\right)\left(\rho f_{D,j}^{-1} + k\right)^2$

[20] An alternative model of exogenous demand (which breaks the no-trade theorem) is that instead of having sentiment shocks, agents could simply have exogenous liquidity or hedging needs (see Dávila and Parlatore (2018), for example). In that case, restricting investment at any frequency would be very bad for them, since trading is fundamentally valuable. The optimal policy for agents of that type would be to subsidize information in order to reduce $a_{2,j}$ towards zero, since that would mean that their liquidity needs did not affect prices (e.g. when forced to buy they would not drive prices up). See the following discussion.

of information that investors acquire becomes infinite, prices become completely informative:

$$\lim_{f_{avg,j}^{-1}\to\infty} a_{1,j} = 1 \tag{52}$$

$$\lim_{f_{avg,j}^{-1}\to\infty} a_{2,j} = 0 \tag{53}$$

and noise traders have zero average losses:

$$\lim_{f_{avg,j}^{-1}\to\infty} E\left[n_j r_j\right] = 0 \tag{54}$$

Increases in $f_{avg,j}^{-1}$ could be encouraged by subsidizing or otherwise encouraging information production by investors (e.g. a tax credit for research), mandating greater disclosure by firms, or by removing barriers to information revelation. In the context of the discussion in the previous section (equation (47)), this corresponds to actively trying to reduce the ψ_j that investors face (ideally to zero, if the goal is to send $f_{avg,j}^{-1}$ to infinity). More generally, if the noise traders are thought of as speculators, these results say that the effect of speculators is reduced when the sophisticated investors have more information.

Note, interestingly, that the optimal policy for helping noise traders and simultaneously increasing price informativeness is the opposite of what we found above would help the long-term investors. The simple reason is that the profits of the long-term investors are earned at the expense of the noise traders.

In the end, therefore, there is a clear tension in the model among short-term investors, long-term investors, and noise traders. Long-term investors benefit from reductions in f_{avg}^{-1} at high frequencies, but that comes at the cost of reducing price informativeness and hurting noise traders and short-term investors. Noise traders benefit from increasing f_{avg}^{-1} (when f_{avg}^{-1} is sufficiently large, at least), but that hurts the informed investors in general, since their trading opportunities shrink. These results follow from the simple fact that these investors are playing a zero-sum game. To those who sit outside the financial market, if what matters most is price efficiency, then obviously a policy encouraging greater information acquisition and higher $f_{avg,j}^{-1}$ will be ideal, all else equal.

5 Quadratic trading costs

The restriction that investors have *exactly* zero exposure at certain frequencies is a natural one to study in the model. But there are other ways of imposing limits on investors' exposures across frequencies. We now examine the equilibrium when there are quadratic costs of trading. Relative to the frictionless benchmark, introducing these costs has analogous effects to the more abstract restriction $q_{i,j} = 0$ for $j \in \mathcal{R}$. Changes in trading costs could be caused either by the imposition of a quadratic tax on shares traded (i.e. a particular form of a Tobin tax), or by changes in the trading technology.

The model does not literally have trade over time. However, the exposures that investors choose in the futures market can be replicated through a commitment to trade (at a fixed price) the fundamental on future dates. That is, define a date-t equity claim to be an asset that pays dividends equal to the fundamental on each date from $t+1$ to T. Since the futures contracts involve exchanging money only at maturity, the date-t cost of an equity claim is $P_t^{equity} = \sum_{j=1}^{T-t} \beta^{-j} P_{t+j}$. An investor's exposure to fundamentals on date t, $Q_{i,t}$ can be acquired either by buying $Q_{i,t}$ units of forwards on date 0 or by holding $Q_{i,t}^{EQ}$ units of equity entering date t. In the latter case, the volume of trade by investor i would be equal to the change in $Q_{i,t}$ over time. That is, $\Delta Q_{i,t}^{EQ} = \Delta Q_{i,t}$.

We assume that investors now maximize the following objective:

$$U_{0,i} = \max_{\{Q_{i,t}\}} E_{0,i}\left[T^{-1}\sum_{t=1}^{T} Q_{i,t}\left(D_t - P_t\right)\right] - \frac{1}{2}cT^{-1}E_{0,i}\left[QV\left\{Q_i\right\}\right] - \frac{1}{2}bT^{-1}E_{0,i}\left[\sum_{t=1}^{T} Q_{i,t}^2\right], \quad (55)$$

where $b > 0$ is a cost of holding large positions in the assets, $c \geq 0$ is a cost incurred from quadratic variation in positions, with quadratic variation defined as:

$$QV\left\{Q_i\right\} \equiv \left[\sum_{t=2}^{T}\left(Q_{i,t} - Q_{i,t-1}\right)^2 + \left(Q_{i,1} - Q_{i,T}\right)^2\right]. \quad (56)$$

The term involving b in (55) replaces the aversion to variance in the benchmark setting. That change is made for the sake of tractability, but its economic consequences are minimal (see, e.g., Kasa, Walker, and Whiteman (2013)). We also set discount rates to zero here to maintain tractability.

Appendix F shows that:

$$T^{-1}QV\left\{Q_i\right\} = 2\sum_{j=1}^{T}\sin^2\left(\omega_{\lfloor j/2\rfloor}/2\right)q_{i,j}^2. \tag{57}$$

Note that we have defined quadratic variation as the sum of the squared changes in $Q_{i,t}$ between $t = 2$ and T plus $(Q_{i,1} - Q_{i,T})^2$. Without the final term, there would be no cost to investors of entering and exiting very large positions at the beginning and end of the investment period. This term helps account for that, and has the added benefit of yielding the simple closed-form expression in the frequency domain reported above. The right-hand side shows that the quadratic variation in the volume induced by an investor depends on their squared exposures at each frequency scaled by $\sin^2\left(\omega_{\lfloor j/2\rfloor}/2\right)$, which rises from 0 to 1 as j rises. Intuitively, when $c > 0$, holding exposure to higher frequency fluctuations in fundamentals is more costly because it requires more frequent portfolio rebalancing.

The equilibrium of the model is described in detail in Appendix F. Here, we highlight key results and explain how they relate to the previous results on restricting trade frequencies.

Result 3 *When $c > 0$, all else equal, investors' equilibrium signal precision is higher at lower frequencies.*

With the assumption of fixed quadratic trading costs, the marginal benefit of increasing precision at frequency j is given by:

$$\frac{1}{2}(c\sin^2\left(\omega_{\lfloor j/2\rfloor}/2\right) + b)^{-1}Var\left[d_j \mid p_j, y_{i,j}\right]^2. \tag{58}$$

In particular, it is declining with both the signal precision and the frequency of exposure. Given that the marginal cost of information is the same across frequencies, investors choose higher signal precisions at lower frequencies, all else equal.

The main result regarding the effect of the quadratic trading cost is the following.

Result 4 *A small increase in trading costs, when starting from zero, reduces information acquisition at all frequencies except frequency 0. The effect is larger at higher frequencies. As a corollary, the effect of an increase in trading costs on price informativeness is weaker at longer horizons.*

The first part of this result suggests that if the goal is to reduce short-term investment, then a quadratic tax is a more blunt instrument than placing an explicit restriction on investment at targeted frequencies. A tax on volume affects all investors, regardless of the strategy that they follow. However, the second part of the result says that trading costs affect short-term strategies most strongly. The quadratic cost thus leads, endogenously, to the same changes in information acquisition studied in the main model; namely, the variance of dividends conditional on prices, $Var(d_j|p_j)$, rises more at higher frequencies. The corollary regarding price informativeness refers to the fact that the variance of moving averages of the form:

$$Var\left(\frac{1}{n}\sum_{m=0}^{n-1} D_{t+m} \mid P\right) \tag{59}$$

increases less as a result of the increase in trading costs for longer horizons n. In the extreme case of $n = T$, which corresponds to the frequency 0 component of the signals, the increase in trading costs has in fact *no* effect on equilibrium signal precision and thus price informativeness. This can be seen from the expression for the marginal benefit of signal precision above, which is independent of c when $j = 0$.

Finally, to examine the effects of trading costs on noise trader profits, we have

Result 5 *Prices continue to take the form*

$$p_j = a_{1,j}d_j + a_{2,j}z_j \tag{60}$$

At all frequencies, increases in trading costs weakly reduce $a_{1,j}$ and strictly increase $a_{2,j}$ (except at frequency zero, where they have no effect).

Again, an increase in trading costs is broadly similar to a restriction on investment in the sense that it makes markets less liquid and prices less informative. By liquid what we mean is that an exogenous demand shock – an increase in z_j – has a larger effect on prices when trading costs are larger. This policy can therefore reduce the losses of noise traders by reducing their overall trade with the informed investors, but again at the cost of less informative prices. As above, if one has evidence that f_Z is large relative to f_D at high frequencies, then this trade-off may be favorable.

There is not much to learn about, so losing information has relatively low costs, and since the sentiment shocks are large, inhibiting them is particularly valuable.

Thus, overall, the message of the model with quadratic costs is consistent with the previous analysis. Increasing trading costs leads to less informed trading and the effect is tilted toward high frequencies; at lower frequencies, information acquisition decisions are less impacted. As a result, the effect of the increase on the informativeness of prices for fundamentals in the long run is limited.

6 Conclusion

The aim of this paper is to understand the effects of policies aimed at reducing "short-termism" in financial markets. It develops results on the effects on price informativeness and investor welfare of restrictions on investment and information acquisition at different frequencies. In order to study those questions, we develop a model in which investors can make meaningful decisions about the horizon of their investment strategies, and in which they face endogenous information choices.

We obtain three main results:

1. Restricting short-term investment affects short-run but not long-run price informativeness and return volatility.

2. Restricting short-term investment hurts both short- and long-term investors, but helps noise traders.

3. Taxing or restricting the availability of short-term information helps long-term investors, hurts short-term investors and noise traders, and reduces short-term price efficiency.

The first result is a natural consequence of the statistical independence of the model across frequencies. The second result shows that while the entry of short-term *investors* reduces the utility and profits of long-term investors, restricting short-term *investment* in response to that entry does not make long-term investors better off. A buy-and-hold investor is able to provide the market short-term liquidity – a person with a price target of $50 should be willing to accommodate transitory demand shocks that drive the price above their target. Short-term investors are better at such liquidity provision; that is why their entry makes long-term investors worse off. But eliminating all short-term investment does not solve the problem. In fact, it makes it worse by eliminating the earnings from liquidity provision for all investors. However, the results for noise traders are reversed

– they benefit from restrictions on investment and are hurt by limits on information.

Finally, the third result shows that information policies have distributional effects. If one's goal is to both maximize price informativeness and limit the impact of speculation by noise traders, subsidizing information acquisition is the correct policy in the model.

We do not make normative claims about what the right objective is. There are many externalities not considered here. For example, price informativeness is important to many agents in the economy who are not represented in our model. We also have a specific model of noise traders as irrational agents, but the role of noise trader demand in facilitating trade can also be played by agents who simply have exogenous liquidity needs, in which case the optimal policy response would more clearly tilt towards information subsidies. It is also not obvious whether short- or long-term investors should necessarily be supported. The goal of the paper is not to resolve the question of which policy is best, but rather simply to provide a general analysis of the effects of the various policies.

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A Noise trader demand

We assume that noise traders have preferences similar to those of sophisticates, but they have different information. They receive signals about fundamentals, and believe that the signals are informative, although the signals are actually random. The signals are also perfectly correlated across the noise traders, so that they do not wash out in the aggregate. They can be therefore thought of as common sentiment shocks among noise traders. Furthermore, the noise traders assume that prices contain no information about fundamentals.

The noise traders optimize

$$\max_{\{N_t\}_{t=1}^{T}} T^{-1}\sum_{t=1}^{T}\beta^t N_t E_{0,N}\left[D_t - P_t\right] - \frac{1}{2}\left(\rho T\right)^{-1} Var_{0,N}\left[\sum_{t=1}^{T}\beta^t N_t\left(D_t - P_t\right)\right] \tag{61}$$

where N_t is the demand of the noise traders and $E_{0,N}$ and $Var_{0,N}$ are their expectation and variance operators conditional on their signals.

We model the noise traders as being Bayesians who simply misunderstand the informativeness of their signals, and ignore prices. Their prior belief, before receiving signals, is that

$$D \sim N\left(0, \Sigma_N^{prior}\right). \tag{62}$$

They then receive signals that they believe (incorrectly) are of the form

$$S \sim N\left(D, \Sigma_N^{signal}\right). \tag{63}$$

The usual Bayesian update then yields the distribution of D conditional on S,

$$D \quad | \quad S \sim N\left(\Sigma_N \left(\Sigma_N^{signal}\right)^{-1} S, \Sigma_N\right) \tag{64}$$

$$\text{where } \Sigma_N \equiv \left(\left(\Sigma_N^{signal}\right)^{-1} + \left(\Sigma_N^{prior}\right)^{-1}\right)^{-1}. \tag{65}$$

So we have

$$E_{0,N}[D] = \Sigma_N \left(\Sigma_N^{signal}\right)^{-1} S \tag{66}$$

$$Var_{0,N}[D] = \Sigma_N \tag{67}$$

Define $\tilde{N}_t \equiv \beta^t N_t$ and $\tilde{N} = [N_1, ..., N_T]'$. The optimization problem then becomes

$$\max_{\tilde{N}} T^{-1} \tilde{N}' \left(\Sigma_N \left(\Sigma_N^{signal}\right)^{-1} S - P\right) - \frac{1}{2} (\rho T)^{-1} \tilde{N}' \Sigma_N \tilde{N}. \tag{68}$$

This has the solution:

$$\tilde{N} = \rho^{-1} \Sigma_N^{-1} \left(\Sigma_N \left(\Sigma_N^{signal}\right)^{-1} S - P\right) \tag{69}$$

$$= \rho^{-1} \left(\left(\Sigma_N^{signal}\right)^{-1} S - \Sigma_N^{-1} P\right). \tag{70}$$

For the sake of simplicity, we assume that $\Sigma_N = k^{-1} I$, where I is the identity matrix and k is a parameter. (This can be obtained, for instance, by assuming that $\Sigma_N^{signal} = \Sigma_N^{prior} = 2kI$). We then have

$$\tilde{N} = \rho^{-1} \left(\Sigma_N^{signal} \right)^{-1} S - kP, \tag{71}$$

so that the vector $Z = (Z_1, ..., Z_T)'$ from the main text is:

$$Z \equiv \rho^{-1} \left(\Sigma_N^{signal} \right)^{-1} S, \tag{72}$$

and the true variance of S, Σ_S, can always be chosen to yield any particular $\Sigma_Z \equiv Var(Z)$ by setting

$$\Sigma_S = \rho^2 \Sigma_N^{signal} \Sigma_Z \Sigma_N^{signal}. \tag{73}$$

B Time horizon and investment

At first glance, the assumption of mean-variance utility over cumulative returns over a long period of time ($T \to \infty$) may appear to give investors an incentive to primarily worry about long-horizon performance, whereas a small value of T would make investors more concerned about short-term performance. In the present setting, that intuition is not correct – the $T \to \infty$ limit determines how detailed investment strategies may be, rather than incentivizing certain types of strategies.

The easiest way to see why the time horizon controls only the detail of the investment strategies is to consider settings in which T is a power of 2 . If $T = 2^k$, then the set of fundamental frequencies is

$$\left\{ 2\pi j / 2^k \right\}_{j=0}^{2^{k-1}} \tag{74}$$

For $T = 2^{k-1}$, the set of frequencies is

$$\left\{ 2\pi j / 2^{k-1} \right\}_{j=0}^{2^{k-2}} = \left\{ 2\pi \left(2j \right) / 2^k \right\}_{j=0}^{2^{k-2}} \tag{75}$$

That is, when T falls from 2^k to 2^{k-1}, the effect is to simply eliminate alternate frequencies. Reducing T does not change the lowest or highest available frequencies (which are always 0 and π, respectively). It just discretizes the $[0, \pi]$ interval more coarsely; or, equivalently, it means that the matrix Λ is constructed from a smaller set of basis vectors.

When T is smaller – there are fewer available basis functions – Q and its frequency domain analog $q \equiv \Lambda' Q$ have fewer degrees of freedom and hence must be less detailed. So the effect of a small value of T is to make it more difficult for an investor to *isolate* particularly short or long run fluctuations in fundamentals (or any other narrow frequency range). But in no way does T cause the investor's portfolio to depend more on one set of frequencies than another.

C Results on the frequency solution

C.1 Proof of lemma 1

The proof here follows "Time Series Analysis" lecture notes of Suhasini Subba Rao. The broad idea of the proof is as follows. Let Σ be any matrix of the form:

$$\Sigma = \begin{pmatrix} \sigma_0 & \sigma_1 & \dots & \dots & \sigma_{T-1} \\ \sigma_1 & \sigma_0 & \sigma_1 & \dots & \sigma_{T-2} \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_{T-1} & \dots & \dots & \dots & \sigma_0 \end{pmatrix} \tag{76}$$

where $x_0 > 0$. Matrices of this type contain all the variance-covariance matrices of order T of arbitrary weakly stationary processes. The lemma follows from "approximating" Σ by the circulant matrix:

$$\Sigma_{circ} = circ(\sigma_{circ}) \quad , \quad \sigma \equiv (\sigma_0, \sigma_1 + \sigma_{T-1}, \sigma_2 + \sigma_{T-2}, ..., \sigma_{T-2} + \sigma_2, \sigma_{T-1} + \sigma_1)' \, , \tag{77}$$

where, for any real vector $\{x_i\}_{i=0}^{T-1}$,

$$circ(x) \equiv \begin{pmatrix} x_0 & \cdot & \cdot & x_{T-1} \\ x_{T-1} & x_0 & \cdot & x_{T-2} \\ \cdot & & & \\ x_1 & \cdot & \cdot & x_0 \end{pmatrix} . \tag{78}$$

In order to obtain this approximation, we first need the following result.

Appendix lemma 4 *For any matrix Σ of the form given above, and associated circulant matrix Σ_{circ}, the family of vectors Λ defined in the main text exactly diagonalizes Σ_{circ}:*

$$\Sigma_{circ}\Lambda = \Lambda diag\left(\left\{f_{\Sigma}\left(\omega_{\lfloor j/2\rfloor}\right)\right\}_{j=1}^{T}\right), \tag{79}$$

where each distinct eigenvalue in $\left\{f_{\Sigma}\left(\omega_{\lfloor j/2\rfloor}\right)\right\}_{j=1}^{T}$ is given by:

$$f_{\Sigma}(\omega_h) = \sigma_0 + 2\sum_{t=1}^{T-1}\sigma_t cos(\omega_h t), \quad \omega_h \equiv 2\pi h/T, \tag{80}$$

for some $h = 0, ..., \frac{T}{2}$.

Given that Λ is orthonormal,

$$\Lambda'\Sigma_{circ}\Lambda = diag\left(f_{\Sigma}\right). \tag{81}$$

The approximate diagonalization of the matrix Σ consists in writing:

$$\Lambda'\Sigma\Lambda = diag\left(f_{\Sigma}\right) + R_{\Sigma}, \tag{82}$$

where the $T \times T$ matrix R_{Σ} is given by:

$$R_{\Sigma} \equiv \Lambda'\left(\Sigma - \Sigma_{circ}\right)\Lambda. \tag{83}$$

This is an approximation in the sense that R_{Σ} is generically small. Specifically, it is of order T^{-1} element-wise. The following lemma proves the first result stated in lemma 1 of the main text.

Appendix lemma 5 *For any $T \geq 2$, we have:*

$$|R_{\Sigma}| \leq \frac{4}{\sqrt{T}}\sum_{j=1}^{T-1}|j\sigma_j|, \tag{84}$$

where $|M|$ denotes the weak matrix norm, as in the main text.

Proof. Define $\Delta\Sigma = \Sigma_{circc} - \Sigma$. First note that since:

$$\Sigma^{(i,j)} = \begin{cases} \sigma_0 & \text{if } i = j \\ \sigma_{|i-j|} & \text{otherwise} \end{cases}, \tag{85}$$

$$\Sigma^{(i,j)}_{circ} = \begin{cases} \sigma_0 & \text{if } i = j \\ \sigma_{|i-j|} + \sigma_{T-|i-j|} & \text{otherwise} \end{cases}, \tag{86}$$

we have:

$$\Delta\Sigma^{(i,j)} = \begin{cases} 0 & \text{if } i = j \\ \sigma_{T-|i-j|} & \text{otherwise} \end{cases} \tag{87}$$

where $\Sigma^{(i,j)}$ is the (i,j) element of Σ. This means that the matrix $\Delta\Sigma$ has constant and symmetric diagonals. Moreover, the first subdiagonals both contain σ_{T-1}, the second contain σ_{T-2}, and so on. That is,

$$\Delta\Sigma = \begin{pmatrix} 0 & \sigma_{T-1} & \sigma_{T-2} & & \sigma_2 & \sigma_1 \\ \sigma_{T-1} & \ddots & \ddots & \ddots & & \sigma_2 \\ \sigma_{T-2} & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \sigma_{T-2} \\ \sigma_2 & & \ddots & \ddots & \ddots & \sigma_{T-1} \\ \sigma_1 & \sigma_2 & & \sigma_{T-2} & \sigma_{T-1} & 0 \end{pmatrix} \tag{88}$$

Therefore,

$$\sum_{i=1}^{T}\sum_{j=1}^{T} |\Delta\sigma_{i,j}| = 2\sum_{j=1}^{T-1} |j\sigma_j|. \tag{89}$$

Let λ_k denote the k-th column of the matrix Λ. For any $(l, m) \in [1, T]^2$, we have:

$$\begin{aligned}
\left|R_{\Sigma}^{(l,m)}\right| &= |\lambda_l' \Delta\Sigma \lambda_m| \\
&= \left|\sum_{i=1}^{T}\sum_{j=1}^{T} \lambda_{i,l}\lambda_{j,m}\Delta\sigma_{i,j}\right| \\
&\leq \sum_{i=1}^{T}\sum_{j=1}^{T} |\lambda_{i,l}|\,|\lambda_{j,m}|\,|\Delta\sigma_{i,j}| \\
&\leq \sum_{i=1}^{T}\sum_{j=1}^{T} \frac{\sqrt{2}}{\sqrt{T}}\frac{\sqrt{2}}{\sqrt{T}}|\Delta\sigma_{i,j}| \\
&= \tfrac{4}{T}\sum_{j=1}^{T-1} |j\sigma_j|.
\end{aligned} \tag{90}$$

This implies that:

$$||R_{\Sigma}||_{\infty} \leq \frac{4}{T}\sum_{j=1}^{T-1} |j\sigma_j|, \tag{91}$$

where $||.||_{\infty}$ is the element-wise max norm. The inequality for the weak norm follows from the fact that the weak norm and the element-wise max norm satisfy $|.| \leq \sqrt{T}\,||.||_{\infty}$. ■

C.2 Derivation of solution 1

To save notation, we suppress the j subscripts indicating frequencies in this section when they are not necessary for clarity. So in this section f_D, for example, is a scalar representing the spectral density of fundamentals at some arbitrary frequency (rather than vectors, which is what the unsubscripted variables represent in the main text).

C.2.1 Statistical inference

We guess that prices take the form

$$p = a_1 d + a_2 z \tag{92}$$

The joint distribution of fundamentals, signals, and prices is then

$$\begin{bmatrix} d \\ y_i \\ p \end{bmatrix} \sim N\left(0, \begin{bmatrix} f_D & f_D & a_1 f_D \\ f_D & f_D + f_i & a_1 f_D \\ a_1 f_D & a_1 f_D & a_1^2 f_D + a_2^2 f_Z \end{bmatrix}\right) \tag{93}$$

The expectation of fundamentals conditional on the signal and price is

$$\begin{aligned} E\left[d \mid y_i, p\right] &= \begin{bmatrix} f_D & a_1 f_D \end{bmatrix} \begin{bmatrix} f_D + f_i & a_1 f_D \\ a_1 f_D & a_1^2 f_D + a_2^2 f_Z \end{bmatrix}^{-1} \begin{bmatrix} y_i \\ p \end{bmatrix} &(94)\\ &= [1, a_1] \begin{bmatrix} 1 + f_i f_D^{-1} & a_1 \\ a_1 & a_1^2 + a_2^2 f_Z f_D^{-1} \end{bmatrix}^{-1} \begin{bmatrix} y_i \\ p \end{bmatrix} &(95)\end{aligned}$$

and the variance satisfies

$$\begin{aligned} \tau_i &\equiv Var\left[d \mid y_i, p\right]^{-1} = f_D^{-1}\left(1 - \begin{bmatrix} 1 & a_1 \end{bmatrix} \begin{bmatrix} 1 + f_i f_D^{-1} & a_1 \\ a_1 & a_1^2 + a_2^2 f_Z f_D^{-1} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ a_1 \end{bmatrix}\right)^{-1} &(96)\\ &= \frac{a_1^2}{a_2^2} f_Z^{-1} + f_i^{-1} + f_D^{-1} &(97)\end{aligned}$$

We use the notation τ to denote a posterior precision, while f^{-1} denotes a prior precision of one of the basic variables of the model. The above then implies that

$$E\left[d \mid y_i, p\right] = \tau_i^{-1}\left(f_i^{-1} y_i + \frac{a_1}{a_2^2} f_Z^{-1} p\right) \tag{98}$$

C.2.2 Demand and equilibrium

The agent's utility function is (where variables without subscripts here indicate vectors),

$$\begin{aligned} U_i &= \max_{\{Q_{i,t}\}} \rho^{-1} E_{0,i}\left[T^{-1} \tilde{Q}_i' (D - P)\right] - \frac{1}{2}\rho^{-2} Var_{0,i}\left[T^{-1/2} \tilde{Q}_i' (D - P)\right] &(99)\\ &= \max_{\{Q_{i,t}\}} \rho^{-1} E_{0,i}\left[T^{-1} \tilde{q}_i' (d - p)\right] - \frac{1}{2}\rho^{-2} Var_{0,i}\left[T^{-1/2} \tilde{q}_i' (d - p)\right] &(100)\\ &= \max_{\{Q_{i,t}\}} \rho^{-1} T^{-1} \sum_j \tilde{q}_{i,j} E_{0,i}\left[(d_j - p_j)\right] - \frac{1}{2}\rho^{-2} T^{-1} \sum_j \tilde{q}_{i,j}^2 Var_{0,i}\left[d_j - p_j\right], &(101)\end{aligned}$$

where the last line follows by imposing the asymptotic independence of d across frequencies (we analyze the error induced by that approximation below). The utility function is thus entirely separable across frequencies, with the optimization problem for each $\tilde{q}_{i,j}$ independent from all others.

Taking the first-order condition associated with the last line above for a single frequency (with $\tilde{q}_i$, d, etc. again representing scalars, for any j), we obtain

$$\begin{aligned} \tilde{q}_i &= \rho\tau_i E\left[d-p \mid y_i, p\right] && (102) \\ &= \rho\left(f_i^{-1}y_i + \left(\frac{a_1}{a_2^2}f_Z^{-1} - \tau_i\right)p\right) && (103) \end{aligned}$$

Summing up all demands and inserting the guess for the price yields

$$\begin{aligned} -z + k\left(a_1 d + a_2 z\right) &= \int_i \rho\left(f_i^{-1}y_i + \left(\frac{a_1}{a_2^2}f_Z^{-1} - \tau_i\right)\left(a_1 d + a_2 z\right)\right)di && (104) \\ &= \int_i \rho\left(f_i^{-1}d + \left(\frac{a_1}{a_2^2}f_Z^{-1} - \tau_i\right)\left(a_1 d + a_2 z\right)\right)di, && (105) \end{aligned}$$

where the second line uses the law of large numbers. Matching coefficients on d and z then yields

$$\begin{aligned} \int_i \rho\left(\frac{a_1}{a_2^2}f_Z^{-1} - \tau_i\right)di &= -a_2^{-1}\left(1-ka_2\right) && (106) \\ \int_i \rho f_i^{-1}a_1^{-1} + \rho\left(\frac{a_1}{a_2^2}f_Z^{-1} - \tau_i\right)di &= k && (107) \end{aligned}$$

and therefore

$$\begin{aligned} k - \int_i \rho f_i^{-1}a_1^{-1} &= a_2^{-1}\left(ka_2 - 1\right) && (108) \\ \int_i \rho f_i^{-1} &= \frac{a_1}{a_2} && (109) \end{aligned}$$

Now define aggregate precision to be

$$f_{avg}^{-1} \equiv \int_i f_i^{-1}di \qquad (110)$$

We then have

$$\tau_i = \frac{a_1^2}{a_2^2} f_Z^{-1} + f_i^{-1} + f_D^{-1} \tag{111}$$

$$\tau_{avg} \equiv \int \tau_i di = \left(\rho f_{avg}^{-1}\right)^2 f_Z^{-1} + f_{avg}^{-1} + f_D^{-1} \tag{112}$$

Inserting the expression for τ_i into (106) yields

$$a_1 = \frac{\tau_{avg} - f_D^{-1}}{\tau_{avg} + \rho^{-1} k} \tag{113}$$

$$a_2 = \frac{a_1}{\rho f_{avg}^{-1}} \tag{114}$$

The expression for a_1 can be written more explicitly as:

$$a_1 = \frac{\tau_{avg} - f_D^{-1}}{\tau_{avg} + \rho^{-1} k} = \frac{\frac{a_1^2}{a_2^2} f_Z^{-1} + f_{avg}^{-1} + f_D^{-1} + \rho^{-1} k - \rho^{-1} k - f_D^{-1}}{\frac{a_1^2}{a_2^2} f_Z^{-1} + f_{avg}^{-1} + f_D^{-1} + \rho^{-1} k} \tag{115}$$

$$= 1 - \frac{\rho^{-1} k + f_D^{-1}}{\left(\rho f_{avg}^{-1}\right)^2 f_Z^{-1} + f_{avg}^{-1} + \rho^{-1} k + f_D^{-1}}. \tag{116}$$

The expression for a_2 is invalid in the case when $f_{avg}^{-1} = 0$. In that case, we have

$$a_2 = \frac{1}{\rho f_D^{-1} + k}. \tag{117}$$

C.3 Proof of Proposition 1

We use the notation $\bar{O}$ to mean that, for any matrices A and B,

$$|A - B| = \bar{O}\left(T^{-1/2}\right) \Longleftrightarrow |A - B| \leq bT^{-1/2} \tag{118}$$

for some constant b and for all T. This is a stronger statement than typical big-O notation in that it holds for all T, as opposed to holding only for some sufficiently large T. Standard properties of norms yield the following result. If $|A-B|=\bar{O}\left(T^{-1/2}\right)$ and $|C-D|=\bar{O}\left(T^{-1/2}\right)$, then

$$
\begin{aligned}
|cA-cB| &= \bar{O}\left(T^{-1/2}\right) && (119)\\
\left|A^{-1}-B^{-1}\right| &= \bar{O}\left(T^{-1/2}\right) && (120)\\
|(A+C)-(B+D)| &= \bar{O}\left(T^{-1/2}\right) && (121)\\
|AC-BD| &= \bar{O}\left(T^{-1/2}\right) && (122)
\end{aligned}
$$

In other words, convergence in weak norm carries through under addition, multiplication, and inversion. Following the time domain solution (8), A_1 and A_2 can be expressed as a function of the Toeplitz matrices Σ_D, Σ_Z and Σ_{avg} using those operations. it follows that $|A_1-\Lambda diag\left(a_1\right)\Lambda'|\leq c_1T^{-\frac{1}{2}}$ for some constant c_1, and the same holds for A_2 for some constant c_2.

For the variance of prices, we define

$$
\begin{aligned}
R_1 &\equiv A_1-\Lambda diag\left(a_1\right)\Lambda', && (123)\\
R_2 &\equiv A_2-\Lambda diag\left(a_2\right)\Lambda'. && (124)
\end{aligned}
$$

In what follows, we use the strong norm $\|\cdot\|$, defined as:

$$
\|A\|=\max_{x'x=a}\left(x'A'Ax\right)^{\frac{1}{2}}. \tag{125}
$$

Finally, we use the following property of the weak norm: for any two square matrices A, B of size $T\times T$,

$$
|AB|\leq\sqrt{T}\,|A|\,|B|. \tag{126}
$$

The proof for this inequality is standard and reported at the end of this section. We then have the following:

$$
\begin{aligned}
|Var\left[P-\Lambda p\right]| &= \left|Var\left[(A_1-\Lambda a_1\Lambda')D+(A_2-\Lambda a_2\Lambda')Z\right]\right| && (127)\\
&\leq \left|R_1\Sigma_D R_1'\right|+\left|R_2\Sigma_Z R_2'\right| && (128)\\
&\leq \sqrt{T}\left(\left|R_1\Sigma_D\right|\left|R_1\right|+\left|R_2\Sigma_Z\right|\left|R_2\right|\right) && (129)\\
&\leq \sqrt{T}\left(\left\|\Sigma_D\right\|\left|R_1\right|^2+\left\|\Sigma_Z\right\|\left|R_2\right|^2\right) && (130)\\
&\leq \sqrt{T}K\left(\left|R_1\right|^2+\left|R_2\right|^2\right). && (131)
\end{aligned}
$$

The second line follows from the triangle inequality. The third line comes from property (126). The fourth line uses the fact that for any two square matrices G, H, $||GH|| \leq \|G\|\,|H|$; for a proof, see Gray (2006), lemma 2.3. The last line follows from the assumption that the eigenvalues of Σ_D and Σ_Z are bounded. Indeed, since Σ_D and Σ_Z are symmetric and real, they are Hermitian; following Gray (2006), eq. (2.16), we then have $\|\Sigma_Z\| = \max_t |\alpha_{Z,t}|$ and $\|\Sigma_D\| = \max_t |\alpha_{D,t}|$, where $\alpha_{X,t}$ denotes the eigenvalues of the matrix X.

Given that $|R_1| \leq c_1 T^{-\frac{1}{2}}$ and $|R_2| \leq c_2 T^{-\frac{1}{2}}$, this implies:

$$
\begin{aligned}
|Var\left[P-\Lambda p\right]| &\leq K\sqrt{T}\left(c_1^2+c_2^2\right)T^{-1} && (132)\\
&= c_P T^{-\frac{1}{2}}. && (133)
\end{aligned}
$$

A similar proof establishes the result for $\left|Var\left[\tilde{Q}-\Lambda\tilde{q}\right]\right|$.

To prove inequality (126), note that:

$$
\begin{aligned}
|AB|^2 &= \tfrac{1}{T}\sum_{m,n}\left(\sum_{t=1}^{T}a_{mt}b_{tn}\right)^2\\
&\leq \tfrac{1}{T}\sum_{m,n}\left(\sum_{t=1}^{T}a_{mt}^2\right)\left(\sum_{t=1}^{T}b_{tn}^2\right)\\
&= \tfrac{1}{T}\left(\sum_{m,t}a_{mt}^2\right)\left(\sum_{n,t}b_{nt}^2\right)\\
&= T\left(\tfrac{1}{T}\left(\sum_{m,t}a_{mt}^2\right)\right)\left(\tfrac{1}{T}\left(\sum_{n,t}b_{nt}^2\right)\right)\\
&= T\left|A\right|^2\left|B\right|^2,
\end{aligned}
\qquad (134)
$$

so that $|AB| \leq \sqrt{T}\,|A|\,|B|$. In this sequence of inequalities, going from the second to the third line uses the Cauchy-Schwarz inequality.

C.4 Proof of lemma 2

First, since the trace operator is invariant under rotations,

$$tr\left(\Sigma_i^{-1}\right) = \sum_j f_{i,j}^{-1}. \tag{135}$$

The information constraint is linear in the frequency-specific precisions. Investors also face a technical constraint that the elements of $f_{i,j}$ corresponding to paired sines and cosines must have the same value. That is, if $\lfloor j/2 \rfloor = \lfloor k/2 \rfloor$, then $f_{i,j} = f_{i,k}$; this condition is necessary for $\varepsilon_{i,t}$ to be stationary.

Inserting the optimal value of $q_{i,j}$ into the utility function, we obtain

$$E_{-1}\left[U_{i,0}\right] \equiv \frac{1}{2} E\left[T^{-1} \sum_j \tau_{i,j} E\left[d_j - p_j \mid y_{i,j}, p_j\right]^2\right] \tag{136}$$

$U_{i,0}$ is utility conditional on an observed set of signals and prices. $E_{-1}\left[U_{i,0}\right]$ is then the expectation taken over the distributions of prices and signals.

$Var\left[E\left[d_j - p_j \mid y_{i,j}, p_j\right]\right]$ is the variance of the part of the return on portfolio j explained by $y_{i,j}$ and p_j, while $\tau_{i,j}^{-1}$ is the residual variance. The law of total variance says

$$Var\left[d_j - p_j\right] = Var\left[E\left[d_j - p_j \mid y_{i,j}, p_j\right]\right] + E\left[Var\left[d_j - p_j \mid y_{i,j}, p_j\right]\right] \tag{137}$$

where the second term on the right-hand side is just $\tau_{i,j}^{-1}$ and the first term is $E\left[E\left[d_j - p_j \mid y_{i,j}, p_j\right]^2\right]$ since everything has zero mean. The unconditional variance of returns is

$$Var(r_j) = Var\left[d_j - p_j\right] = \left(1 - a_{1,j}\right)^2 f_{D,j} + \frac{a_{1,j}^2}{\left(\rho f_{avg,j}^{-1}\right)^2} f_{Z,j}. \tag{138}$$

So then

$$E_{-1}\left[U_{i,0}\right]=\frac{1}{2}T^{-1}\sum_{j}\left[\left(\left(1-a_{1,j}\right)^{2}f_{D,j}+\frac{a_{1,j}^{2}}{\left(\rho f_{avg}^{-1}\right)^{2}}f_{Z,j}\right)\tau_{i,j}-1\right]. \tag{139}$$

We thus obtain the result that agent i's expected utility is linear in the precision of the signals that they receive (since $\tau_{i,j}$ is linear in $f_{i,j}^{-1}$; see appendix equation 111). Now define

$$\lambda_{j}\left(f_{avg,j}^{-1}\right)\equiv(1-a_{1,j})^{2}f_{D,j}+\left(\frac{a_{1,j}}{\rho f_{avg,j}^{-1}}\right)^{2}f_{Z,j}=Var(r_{j}). \tag{140}$$

From equations (112)-(113), λ_j can be re-written as:

$$\lambda_{j}\left(f_{avg,j}^{-1}\right)=\frac{f_{D,j}\left(f_{D,j}^{-1}+\rho^{-1}k\right)^{2}+(\rho f_{avg,j}^{-1})^{2}f_{Z,j}^{-1}+f_{Z,j}\rho^{-2}}{\left((\rho f_{avg,j}^{-1})^{2}f_{Z,j}^{-1}+f_{D,j}^{-1}+\rho^{-1}k+f_{avg,j}^{-1}\right)^{2}}, \tag{141}$$

which can be further decomposed as:

$$\begin{aligned}\lambda_{j}\left(f_{avg,j}^{-1}\right) &= \frac{1}{\left((\rho f_{avg,j}^{-1})^{2}f_{Z,j}^{-1}+f_{D,j}^{-1}+\rho^{-1}k+f_{avg,j}^{-1}\right)^{2}}\\ &+ \frac{f_{Z,j}-\frac{f_{avg,j}^{-1}}{\rho}}{\left((\rho f_{avg,j}^{-1})^{2}f_{Z,j}^{-1}+f_{D,j}^{-1}+\rho^{-1}k+f_{avg,j}^{-1}\right)^{2}}\\ &+ \frac{\rho^{-1}k(1+f_{D,j}^{-1}\rho^{-1}k)}{\left((\rho f_{avg,j}^{-1})^{2}f_{Z,j}^{-1}+f_{D,j}^{-1}+\rho^{-1}k+f_{avg,j}^{-1}\right)^{2}}\end{aligned} \tag{142}$$

Each of these three terms is decreasing in $f_{avg,j}^{-1}$, so that the function $\lambda_j(\cdot)$ is decreasing.

D Results on price informativeness with restricted frequencies

D.1 Result 1 and corollaries 1.1 and 1.5

When there are no active investors and just exogenous supply, we have that $0=z_j+kp_j$ and so:

$$\begin{aligned} p_{j} &= k^{-1}z_{j}, \qquad (143)\\ r_{j} &= d_{j}-k^{-1}z_{j}. \qquad (144)\end{aligned}$$

Because of the separability of information choices across frequencies, the coefficients $a_{1,j}$ and $a_{2,j}$ are unchanged at all other frequencies. Moreover, it is clear that $Var(d_j|p_j)=Var(d_j)$ at the restricted frequencies, since prices now only carry information about supply, which is uncorrelated with dividends.

Note that for any $j \in \mathcal{R}$,

$$Var(r_j) = f_{D,j} + \frac{f_{Z,j}}{k^2}. \tag{145}$$

Additionally, if investors were allowed to hold *exposure* at those frequencies, but the endogenously chose not to allocate any *attention* to the frequency, the return volatility would be:

$$Var_{unrestr.}(r_j) = \lambda_j(0) = f_{D,j} + \frac{f_{Z,j}}{\left(k + \rho f_{D,j}^{-1}\right)^2} < Var(r_j). \tag{146}$$

D.2 Corollary 1.2 and result 1.4

Under the diagonal approximation, we have:

$$D \mid P \sim N\left(\bar{D}, \Lambda diag\left(\tau_0^{-1}\right)\Lambda'\right) \tag{147}$$

where τ_0 is a vector of frequency-specific precisions conditional on prices, as of time 0. Given the independence of prices across frequencies, the j-th element of τ_0 is:

$$\tau_{0,j}^{-1} = Var(d_j \mid p_j). \tag{148}$$

Using this expression, we can compute:

$$\begin{aligned}
Var\left(D_t \mid P\right) &= 1_t'\Lambda diag\left(\tau_0^{-1}\right)\Lambda' 1_t & (149)\\
&= \left(\Lambda' 1_t\right)' diag\left(\tau_0^{-1}\right)\left(\Lambda' 1_t\right) & (150)\\
&= \sum_j \lambda_{t,j}^2 Var(d_j \mid p_j) & (151)\\
&= \lambda_{t,0}^2 Var(d_0|p_0) + \lambda_{t,\frac{T}{2}}^2 Var(d_{\frac{T}{2}} \mid p_{\frac{T}{2}}) + \sum_{k=1}^{T/2-1}\left(\lambda_{t,2k}^2 + \lambda_{t,2k+1}^2\right) Var(d_k \mid p_k) & (152)
\end{aligned}$$

where 1_t is a vector equal to 1 in its t-th element and zero elsewhere, and $\lambda_{t,j}$ is the t, j element of Λ. The last line follows from the fact that all the spectra have $f_{X,2k} = f_{X,2k+1}$ for $0 < k < T/2 - 1$.

Furthermore, note that for $0 < k < T/2 - 1$,

$$\begin{aligned} \lambda_{t,2k}^2 + \lambda_{t,2k+1}^2 &= \frac{2}{T}\cos\left(\omega_k\left(t-1\right)\right)^2 + \frac{2}{T}\sin\left(\omega_k\left(t-1\right)\right)^2 && (153) \\ &= \frac{2}{T} && (154) \end{aligned}$$

which yields equation (34). Result 3 immediately follows from this expression.

Result 1.4 uses the fact that

$$\begin{aligned} Var\left(D_t - D_{t-1} \mid P\right) &= \left(\lambda_{t,1} - \lambda_{t-1,1}\right)^2 \tau_{0,1}^{-1} + \left(\lambda_{t,T} - \lambda_{t-1,T}\right)^2 \tau_{0,\frac{T}{2}}^{-1} && (155) \\ &+ \sum_{k=1}^{T/2-1}\left[\left(\lambda_{t,2k} - \lambda_{t-1,2k}\right)^2 + \left(\lambda_{t,2k+1} - \lambda_{t-1,2k+1}\right)^2\right]\tau_{0,k}^{-1}, && (156) \end{aligned}$$

and the fact that $(\cos(x) - \cos(y))^2 + (\sin(x) - \sin(y))^2 = 4\sin\left(\frac{1}{2}(x-y)\right)^2 = 2\left(1 - \cos(x-y)\right)$.

E Results on investor outcomes

E.1 Result 2

Expression (38) in the main text follows from the steps used in appendix D.2. Recall from (103) that, omitting the j notation,

$$\begin{aligned} \tilde{q}_i &= \rho\left(f_i^{-1} y_i + \left(\frac{a_1}{a_2^2} f_Z^{-1} - \tau_i\right) p\right) && (157) \\ &= \rho f_i^{-1}\varepsilon_i + \rho\left(f_i^{-1} + \left(\frac{a_1}{a_2^2} f_Z^{-1} - \tau_i\right) a_1\right) d + \rho\left(\frac{a_1}{a_2^2} f_Z^{-1} - \tau_i\right) a_2 z && (158) \end{aligned}$$

Recall also that:

$$\tau_i = \left(\frac{a_1}{a_2}\right)^2 f_Z^{-1} + f_D^{-1} + f_i^{-1}, \tag{159}$$

so that:

$$\tilde{q}_i = \rho\left(\tau_i - \left(\frac{a_1}{a_2}\right)^2 f_Z^{-1} - f_D^{-1}\right)\varepsilon_i + \rho\left(f_i^{-1} + \left(\frac{a_1}{a_2^2} f_Z^{-1} - \tau_i\right) a_1\right) d + \rho\left(\frac{a_1}{a_2^2} f_Z^{-1} - \tau_i\right) a_2 z \tag{160}$$

Moreover,

$$f_i^{-1} - a_1\tau_i + \left(\frac{a_1}{a_2}\right)^2 f_Z^{-1} = \tau_i - \left(\frac{a_1}{a_2}\right)^2 f_Z^{-1} - f_D^{-1} - a_1\tau_i + \left(\frac{a_1}{a_2}\right)^2 f_Z^{-1} \tag{161}$$

$$= (1-a_1)\tau_i - f_D^{-1}. \tag{162}$$

Therefore

$$\rho^{-1}\tilde{q}_i = \left(\tau_i - \left(\frac{a_1}{a_2}\right)^2 f_Z^{-1} - f_D^{-1}\right)\varepsilon_i + \left((1-a_1)\tau_i - f_D^{-1}\right)d + \left(\frac{a_1}{a_2} f_Z^{-1} - a_2\tau_i\right)z, \tag{163}$$

so that

$$\rho^{-2}Var\left(\tilde{q}_i\right) = \left(\tau_i - \left(\frac{a_1}{a_2}\right)^2 f_Z^{-1} - f_D^{-1}\right) + \left((1-a_1)\tau_i - f_D^{-1}\right)^2 f_D + \left(\frac{a_1}{a_2} f_Z^{-1} - a_2\tau_i\right)^2 f_Z. \tag{164}$$

(where the first term uses the fact that $Var\left(f_i^{-1}\varepsilon_i\right) = f_i^{-1}$). The derivative of this expression with respect to τ_i is:

$$\begin{aligned}
\rho^{-2}\frac{\partial Var\left(\tilde{q}_i\right)}{\partial \tau_i} &= 2\tau_i\left((1-a_1)^2 f_D + a_2^2 f_Z\right) - 1 \\
&\geq 2\left(f_D^{-1} + \left(\tfrac{a_1}{a_2}\right)^2 f_Z^{-1}\right)\left((1-a_1)^2 f_D + a_2^2 f_Z\right) - 1 \\
&= 2\left((1-a_1)^2 + a_2^2 f_Z f_D^{-1} + (1-a_1)^2\left(\tfrac{a_1}{a_2}\right)^2 f_Z^{-1} f_D + a_1^2\right) - 1 \\
&= 2\left(1 - 2a_1(1-a_1) + a_2^2 f_Z f_D^{-1} + (1-a_1)^2\left(\tfrac{a_1}{a_2}\right)^2 f_Z^{-1} f_D\right) - 1 \\
&= 2\left(-2a_1(1-a_1) + a_2^2 f_Z f_D^{-1} + (1-a_1)^2\left(\tfrac{a_1}{a_2}\right)^2 f_Z^{-1} f_D\right) + 1 \\
&= 2\left((1-a_1)\left(\tfrac{a_1}{a_2}\right)(f_Z^{-1} f_D)^{\frac{1}{2}} - a_2(f_Z^{-1} f_D)^{-\frac{1}{2}}\right)^2 + 1 \\
&> 0,
\end{aligned} \tag{165}$$

where to go from the first to the second line, we used the fact that $\tau_i \geq \left(\frac{a_1}{a_2}\right)^2 f_Z^{-1} + f_D^{-1}$, and where we also used the fact that $a_1 \leq 1$. Since τ_i is a monotonic transformation of f_i^{-1}, this establishes equation (39) from the main text.

For result 2, first note that $E_{-1}\left[\tilde{Q}_i' R\right] = E_{-1}\left[\tilde{q}_i' \Lambda' \Lambda r\right] = E_{-1}\left[\tilde{q}_i r\right] = \sum_j E_{-1}\left[\tilde{q}_{i,j} r_j\right]$, where the last equality follows from the diagonal approximation. Moreover, straightforward but tedious algebra shows that:

$$\begin{aligned} f_i^{-1} + \left(\frac{a_1}{a_2^2} f_Z^{-1} - \tau_i\right) a_1 &= \rho(f_i^{-1} - f_{avg}^{-1})(1 - a_1) + k a_1, &(166)\\ \left(\frac{a_1}{a_2^2} f_Z^{-1} - \tau_i\right) a_2 &= -\rho(f_i^{-1} - f_{avg}^{-1}) a_2 + (k a_2 - 1). &(167)\end{aligned}$$

We can use these expressions, and the fact that $r = (1 - a_1)d - a_2 z$ to re-write $\tilde{q}_i$ as:

$$\tilde{q}_i = \rho f_i^{-1} \varepsilon_i + \rho \left(f_i^{-1} - f_{avg}^{-1}\right) r + k a_1 d + (k a_2 - 1) z. \tag{168}$$

Therefore,

$$E_{-1}\left[\tilde{q}_i r\right] = \rho \left(f_i^{-1} - f_{avg}^{-1}\right) Var\left(r\right) + k a_1 E_{-1}\left[rd\right] + \left(k a_2 - 1\right) E_{-1}\left[rz\right], \tag{169}$$

which is the decomposition from result 2.

The result that expected profits are nonnegative is a simple consequence of the investors' objective:

$$\max_{\{\tilde{q}_{i,j}\}} \rho^{-1} T^{-1} \sum_j E_{0,i}\left[\tilde{q}_{i,j}\left(d_j - p_j\right)\right] - \frac{1}{2} \rho^{-2} T^{-1} \sum_j Var_{0,i}\left[\tilde{q}_{i,j}\left(d_j - p_j\right)\right] \tag{170}$$

Since the variance is linear in $\tilde{q}_{i,j}^2$, if $E_{0,i}\left[\tilde{q}_{i,j} r_j\right] < 0$, utility can always be increased by setting $\tilde{q}_{i,j} = 0$ (or, even more, by reversing the sign of $\tilde{q}_{i,j}$). In order for $E_{-1}\left[\tilde{q}_{i,j} r_j\right] = 0$, it must be the case that $Var_{-1,i}\left[E_{0,i}\left[d_j - p_j\right]\right] = 0$, since any deviation of $E_{0,i}\left[d_j - p_j\right]$ will cause the investor to optimally take a position. We have, from above,

$$\begin{aligned} a_1 &= \frac{\tau_{avg} - f_D^{-1}}{\tau_{avg} + \rho^{-1} k} = \frac{\left(\rho f_{avg}^{-1}\right)^2 f_Z^{-1} + f_{avg}^{-1}}{\left(\rho f_{avg}^{-1}\right)^2 f_Z^{-1} + f_{avg}^{-1} + f_D^{-1} + \rho^{-1} k} &(171)\\ a_2 &= \frac{a_1}{\rho f_{avg}^{-1}} &(172)\\ \tau_{avg} &\equiv \left(\rho f_{avg}^{-1}\right)^2 f_Z^{-1} + f_{avg}^{-1} + f_D^{-1} &(173)\end{aligned}$$

The expression for a_2 is invalid in the case when $f_{avg}^{-1}=0$. In that case, we have

$$E\left[d \mid y_i, p\right] = \tau_i^{-1}\left(f_i^{-1}y_i + \frac{a_1}{a_2^2}f_Z^{-1}p\right) \tag{174}$$

$$E\left[d-p \mid y_i, p\right] = \tau_i^{-1}f_i^{-1}y_i + \left(\tau_i^{-1}\frac{a_1}{a_2^2}f_Z^{-1}-1\right)(a_1 d + a_2 z) \tag{175}$$

$$Var\left[E\left[d-p \mid y_i, p\right]\right] = \left(\tau_i^{-1}f_i^{-1} + \left(\tau_i^{-1}\frac{a_1}{a_2^2}f_Z^{-1}-1\right)a_1\right)^2 f_D + \left(\tau_i^{-1}\frac{a_1}{a_2^2}f_Z^{-1}-1\right)^2 a_2^2 f_Z \tag{176}$$

Now first we must have $\tau_i^{-1}\frac{a_1}{a_2^2}f_Z^{-1}-1=0$ in order for the third term to be zero. But if that is true, then for the first term to be zero we must have $f_i^{-1}=0$ (since τ_i^{-1} is always positive). Combining $f_i^{-1}=0$ with $\tau_i^{-1}\frac{a_1}{a_2^2}f_Z^{-1}-1=0$, we obtain

$$f_D^{-1} = \rho f_{avg}^{-1} f_Z^{-1} k. \tag{177}$$

E.2 Corollary 2.1

We drop the notation j for clarity. Assume that long-term investors are initially uninformed about the frequency; then $f_i^{-1}=0$, for all i so:

$$\tau_i = \left(\frac{a_1}{a_2}\right)^2 f_Z^{-1} + f_D^{-1}. \tag{178}$$

Using expression (163), we then have

$$\rho^{-1}\tilde{q}_{LF,i} = \left((1-a_1)\left(\frac{a_1}{a_2}\right)^2 f_Z^{-1} - a_1 f_D^{-1}\right)d + \left(\frac{a_1(1-a_1)}{a_2}f_Z^{-1} - a_2 f_D^{-1}\right)z. \tag{179}$$

Given that $r=(1-a_1)d-a_2 z$ and that z and d are independent,

$$\begin{aligned}
\rho^{-1}E_{-1}\left[\tilde{q}_{LF,i}r\right] &= \left((1-a_1)\left(\tfrac{a_1}{a_2}\right)^2 f_Z^{-1} - a_1 f_D^{-1}\right)(1-a_1)f_D - \left(\tfrac{a_1(1-a_1)}{a_2}f_Z^{-1} - a_2 f_D^{-1}\right)a_2 f_Z \\
&= (1-a_1)^2\left(\tfrac{a_1}{a_2}\right)^2 f_Z^{-1}f_D - 2a_1(1-a_1) + a_2^2 f_Z f_D^{-1} \\
&= \left((1-a_1)\left(\tfrac{a_1}{a_2}\right)(f_Z^{-1}f_D)^{\frac{1}{2}} - a_2(f_Z f_D^{-1})^{\frac{1}{2}}\right)^2
\end{aligned} \tag{180}$$

For any $f_{avg}^{-1} > 0$, where $a_1/a_2 = \rho f_{avg}^{-1}$, the derivative of this expression with respect to f_{avg}^{-1} is

$$\begin{aligned}\rho^{-1}\frac{dE_{-1}[\tilde{q}_{LF,i}r]}{df_{avg}^{-1}} &= 2\left((1-a_1)\left(\tfrac{a_1}{a_2}\right)(f_Z^{-1}f_D)^{\frac{1}{2}} - a_2(f_Z f_D^{-1})^{\frac{1}{2}}\right) \\ &\times\left\{\rho\left[(1-a_1)(f_Z^{-1}f_D)^{\frac{1}{2}} - a_1(f_Z f_D^{-1})^{\frac{1}{2}}\right] - \left[(f_Z^{-1}f_D)^{\frac{1}{2}} + (f_Z f_D^{-1})^{\frac{1}{2}}\right]\rho\frac{\partial a_1}{\partial f_{avg}^{-1}}f_{avg}^{-1}\right\}\end{aligned} \tag{181}$$

Moreover, when $f_{avg}^{-1} > 0$,

$$\frac{\partial a_1}{\partial f_{avg}^{-1}}f_{avg}^{-1} = a_1(1-a_1) + (1-a_1)\frac{(\rho f_{avg}^{-1})^2 f_Z^{-1}}{(\rho f_{avg}^{-1})^2 f_Z^{-1} + f_{avg}^{-1} + f_D^{-1} + \rho^{-1}k}. \tag{182}$$

The following limits follow from the discussion in Appendix C.2.2:

$$\lim_{f_{avg}^{-1}\to 0^+} a_1 = 0, \quad \lim_{f_{avg}^{-1}\to 0} a_2 = \frac{1}{\rho f_D^{-1} + k}. \tag{183}$$

Using these limits and the expressions just derived, we arrive at

$$\lim_{f_{avg}^{-1}\to 0^+}\frac{\partial E_{-1}\left[\tilde{q}_{LF,i}r\right]}{\partial f_{avg}^{-1}} = -2\rho\frac{(f_Z f_D^{-1})^{\frac{1}{2}}(f_Z^{-1}f_D)^{\frac{1}{2}}}{f_D^{-1}+\rho^{-1}k} < 0. \tag{184}$$

Re-introducing the notation j, for the frequency at which entry takes place, we then have

$$\frac{d}{df_{avg,j}^{-1}}E_{-1}\left[\sum_t \tilde{Q}_{LF,t}\left(D_t - P_t\right)\right] = \frac{d}{df_{avg,j}^{-1}}\sum_k E_{-1}\left[\tilde{q}_{LF,k}r_k\right] = \frac{d}{df_{avg,j}^{-1}}E_{-1}\left[\tilde{q}_{LF,j}r_j\right] < 0; \tag{185}$$

that is, all the effect of entry on total profits is concentrated on frequency j, where entry reduces profits, as just established.

For the last result, we again use the frequency separability,

$$\frac{d}{df_{avg,j}^{-1}}E_{-1}\left[U_{LF,0}\right] = \frac{d}{df_{avg,j}^{-1}}E_{-1}\left[u_{LF,0,j}\right], \tag{186}$$

where

$$E_{-1}[u_{LF,0,j}] \equiv \frac{1}{2}T^{-1}\left[\left((1-a_{1,j})^2 f_{D,j} + a_{2,j}^2 f_{Z,j}\right)\tau_{i,j} - 1\right] \tag{187}$$

is the component of utility which fluctuates at frequency j. This latter definition uses expression (139), derived in Appendix C.4. Omitting the j notation for clarity, the derivative of this expression with respect to f_{avg} assuming that $f_i^{-1} = 0$ is:

$$\begin{aligned} 2T\tfrac{dE_{-1}[u_{LF,0}]}{df_{avg}^{-1}} &= \left((1-a_1)^2 f_D + a_1^2(\rho f_{avg}^{-1})^2 f_Z\right) 2\rho^2 f_Z^{-1} f_{avg}^{-1} \\ &+ \left(-2(1-a_1)\tfrac{\partial a_1}{\partial f_{avg}^{-1}} f_D + 2a_1 \tfrac{\partial a_1}{\partial f_{avg}^{-1}} \left(\rho f_{avg}^{-1}\right)^2 f_Z + 2a_1^2\rho^2 f_Z f_{avg}^{-1}\right)\left(\left(\rho f_{avg}^{-1}\right)^2 f_Z^{-1} + f_D^{-1}\right) \end{aligned} \tag{188}$$

Given that:

$$\lim_{f_{avg}^{-1}\to 0^+} a_1 = 0, \tag{189}$$

the only term in this expression for which the limit may not be 0 as $f_{avg}^{-1} \to 0^+$ is:

$$-2(1-a_1)\frac{\partial a_1}{\partial f_{avg}^{-1}} f_D + 2a_1 \frac{\partial a_1}{\partial f_{avg}^{-1}} \rho f_{avg}^{-1} f_Z. \tag{190}$$

However, given equation (182), we have that:

$$\lim_{f_{avg}^{-1}\to 0^+} \frac{\partial a_1}{\partial f_{avg}^{-1}} f_{avg}^{-1} = 0, \tag{191}$$

and so the second term in (190) goes to 0 as $f_{avg}^{-1} \to 0^+$. For the second term, note that, using (182) we have that:

$$\frac{\partial a_1}{\partial f_{avg}^{-1}} = \frac{a_1}{f_{avg}^{-1}} + o(1) = \frac{1 + (\rho f_{avg}^{-1}) f_Z^{-1}}{(\rho f_{avg}^{-1})^2 f_Z^{-1} + f_D^{-1} + f_{avg}^{-1} + \rho^{-1}k} + o(1). \tag{192}$$

Therefore,

$$\lim_{f_{avg}^{-1}\to 0^+} 2T\frac{dE_{-1}[u_{LF,0}]}{df_{avg}^{-1}} = -2\frac{f_D}{f_D^{-1} + \rho^{-1}k} = -2f_D a_2 < 0, \tag{193}$$

which proves the last statement of corollary 2.1.

F Quadratic costs

F.1 Frequency domain expressions for trading costs

Using $Q_i = \Lambda q_i$, each agent's position at time t can be written as

$$Q_{i,t} = \sum_j \begin{bmatrix} q_j \cos(2\pi jt/T) \\ +q_{j'} \sin(2\pi jt/T) \end{bmatrix}. \tag{194}$$

Trading costs are then written in terms of $(Q_{i,t} - Q_{i,t-1})^2$ as:

$$QV\{Q_i\} \equiv \sum_{t=2}^{T} (Q_{i,t} - Q_{i,t-1})^2 + (Q_{i,1} - Q_{i,T})^2. \tag{195}$$

We can write that as

$$QV\{Q_i\} = (DQ)'(DQ) \tag{196}$$

where D is a matrix that generates first differences,

$$D \equiv \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix}. \tag{197}$$

Using again the fact that $Q_i = \Lambda q_i$,

$$QV\{Q_i\} = q'\Lambda' D' D\Lambda q \tag{198}$$

In what follows, we will need to evaluate the matrix $\Lambda' D' D\Lambda$. The m, n element of that matrix is the inner product of the m and n columns of $D\Lambda$. Each column of $D\Lambda$ contains the first difference of the corresponding column of Λ, with the exception of the last element, $(D\Lambda)_{m,T}$, which is equal to $\Lambda_{m,t} - \Lambda_{n,T}$. We have the following standard trigonometric results: for $m \neq n$:

$$\begin{aligned}
\sum_{t=1}^{T} \left(\cos\left(\omega_m t\right) - \cos\left(\omega_m \left(t-1\right)\right)\right)\left(\cos\left(\omega_n t\right) - \cos\left(\omega_n \left(t-1\right)\right)\right) &= 0, && (199)\\
\sum_{t=1}^{T} \left(\cos\left(\omega_m t\right) - \cos\left(\omega_m \left(t-1\right)\right)\right)\left(\sin\left(\omega_n t\right) - \sin\left(\omega_n \left(t-1\right)\right)\right) &= 0, && (200)\\
\sum_{t=1}^{T} \left(\sin\left(\omega_m t\right) - \sin\left(\omega_m \left(t-1\right)\right)\right)\left(\sin\left(\omega_n t\right) - \sin\left(\omega_n \left(t-1\right)\right)\right) &= 0, && (201)
\end{aligned}$$

where recall that $\omega_m = \frac{2\pi m}{T}$, and:

$$\begin{aligned}
\sum_{t=1}^{T} \left(\cos\left(\omega_m t\right) - \cos\left(\omega_m \left(t-1\right)\right)\right)^2 &= 2T\sin^2\left(\omega_m/2\right), && (202)\\
\sum_{t=1}^{T} \left(\sin\left(\omega_m t\right) - \sin\left(\omega_m \left(t-1\right)\right)\right)^2 &= 2T\sin^2\left(\omega_m/2\right), && (203)\\
\sum_{t=1}^{T} \left(\cos\left(\omega_m t\right) - \cos\left(\omega_m \left(t-1\right)\right)\right)\left(\sin\left(\omega_m t\right) - \sin\left(\omega_m \left(t-1\right)\right)\right) &= 0. && (204)
\end{aligned}$$

These results immediately imply that the off-diagonal elements of $\Lambda' D' D \Lambda$ are equal to zero and the jth element of the main diagonal is $2T\sin^2\left(\omega_{\lfloor j/2 \rfloor}/2\right)$.

We then have

$$\begin{aligned}
QV\left\{Q_i\right\} &= q\Lambda' D' D \Lambda q && (205)\\
&= \sum_{j=1}^{T} 2T\sin^2\left(\omega_{\lfloor j/2 \rfloor}/2\right) q_{i,j}^2 && (206)
\end{aligned}$$

Total holding costs can be written as:

$$\sum_{t=1}^{T} Q_t^2 = \sum_{j=1}^{T} q_j^2, \qquad (207)$$

which is just Parseval's theorem.

F.2 Equilibrium of the trading cost model

Throughout the analysis, unless it is necessary, we omit the index j of the particular frequency in order to simplify notation.

F.2.1 Investment and equilibrium

The first-order condition for frequency j is

$$\begin{aligned} 0 &= E\left[d_j - p_j \mid y_{i,j}, p_j\right] - 2c\sin^2\left(\omega_{\lfloor j/2 \rfloor}/2\right) q_j - bq_j && (208) \\ q &= \frac{E\left[d_j - p_j \mid y_{i,j}, p_j\right]}{\gamma_j} && (209) \\ &= \gamma_j^{-1}\tau_i^{-1}\left(f_i^{-1}y_i + \left(\frac{a_1}{a_2^2}f_Z^{-1} - \tau_i\right)p\right) && (210) \end{aligned}$$

where

$$\gamma_j \equiv 2c\sin^2\left(\omega_{\lfloor j/2 \rfloor}/2\right) + b \tag{211}$$

is the marginal cost of q_j. We can then solve for the coefficients a_1 and a_2 as before.

Inserting the formula for the conditional expectation and integrating across investors yields

$$\begin{aligned} \int_i \gamma_j^{-1}\tau_i^{-1}\left(f_i^{-1}y_i + \left(\frac{a_1}{a_2^2}f_Z^{-1} - \tau_i\right)(a_1 d - a_2 z)\right)di &= z_j && (212) \\ \int_i \gamma_j^{-1}\tau_i^{-1}\left(f_i^{-1}d + \left(\frac{a_1}{a_2^2}f_Z^{-1} - \tau_i\right)(a_1 d - a_2 z)\right)di &= z_j && (213) \end{aligned}$$

Matching coefficients then yields

$$\begin{aligned} \int_i \gamma_j^{-1}\tau_i^{-1}\left(\frac{a_1}{a_2^2}f_Z^{-1} - \tau_i\right)di &= -a_2^{-1} && (214) \\ \int_i \gamma_j^{-1}\tau_i^{-1}\left(f_i^{-1} + \left(\frac{a_1}{a_2^2}f_Z^{-1} - \tau_i\right)a_1\right)di &= 0 && (215) \end{aligned}$$

Combining those two equations, we obtain

$$\int_i \gamma_j^{-1}\tau_i^{-1}f_i^{-1}di = \frac{a_1}{a_2} \tag{216}$$

Now put the definition of τ_i into that equation for f_i^{-1}

$$\int_i \gamma_j^{-1}\tau_i^{-1}\left(\tau_i - \frac{a_1^2}{a_2^2}f_Z^{-1} - f_D^{-1}\right)di = \frac{a_1}{a_2} \tag{217}$$

$$\gamma_j^{-1}\int_i 1 - \left(\frac{a_1^2}{a_2^2}f_Z^{-1} - f_D^{-1}\right)\tau_i^{-1}di = \frac{a_1}{a_2} \tag{218}$$

F.2.2 Expected utility

At any particular frequency,

$$U_{i,j} = q_{i,j}E_{0,i}\left[d_j - p_j\right] - \frac{1}{2}q_{i,j}^2 2c\sin^2\left(\omega_{\lfloor j/2\rfloor}/2\right) - \frac{1}{2}bq_{i,j}^2 \tag{219}$$

$$= \frac{1}{2}\frac{E\left[d_j - p_j \mid y_{i,j}, p_j\right]^2}{\gamma_j} \tag{220}$$

Expected utility prior to observing signals is then

$$EU_{i,j} \equiv \frac{1}{2}E\left[\frac{E\left[d_j - p_j \mid y_{i,j}, p_j\right]^2}{\gamma_j}\right] \tag{221}$$

$E\left[E\left[d_j - p_j \mid y_{i,j}, p_j\right]^2\right]$ is the variance of the part of the return on portfolio j explained by $y_{i,j}$ and p_j, while $\tau_{i,j}$ is the residual variance. We know from the law of total variance that

$$Var\left[d_j - p_j\right] = Var\left[E\left[d_j - p_j \mid y_{i,j}, p_j\right]\right] + E\left[Var\left[d_j - p_j \mid y_{i,j}, p_j\right]\right] \tag{222}$$

where the second term on the right-hand side is just $\tau_{i,j}^{-1}$ and the first term is $E\left[E\left[d_j - p_j \mid y_{i,j}, p_j\right]^2\right]$ since everything has zero mean. The unconditional variance of returns is simply

$$Var\left[d_j - p_j\right] = Var\left[\left(1 - a_1\right)d_j + a_2 z_j\right] \tag{223}$$

$$= \left(1 - a_{1,j}\right)^2 f_{D,j} + a_2^2 f_{Z,j} \tag{224}$$

So then

$$EU_{i,j} = \frac{1}{2}\frac{Var\left[d_j - p_j\right] - \tau_{i,j}^{-1}}{\gamma_j} \tag{225}$$

What we end up with is that utility is decreasing in $\tau_{i,j}^{-1}$. That is,

$$EU_{i,j} = -\frac{1}{2}\frac{\tau_{i,j}^{-1}}{\gamma_j} + \text{constants}. \tag{226}$$

F.2.3 Information choice

With the linear cost on precision, agents maximize

$$\begin{aligned} & -\frac{1}{2}\frac{\tau_{i,j}^{-1}}{\gamma_j} - \psi f_{i,j}^{-1} && (227)\\ = \quad & -\frac{1}{2}\left(\frac{a_1^2}{a_2^2} f_{Z,j}^{-1} + f_{i,j}^{-1} + f_{D,j}^{-1}\right)^{-1} \gamma_j^{-1} - \psi f_{i,j}^{-1} && (228)\end{aligned}$$

The FOC for $f_{i,j}^{-1}$ is

$$\begin{aligned} \psi &= \frac{1}{2}\tau_{i,j}^{-2}\gamma_j^{-1} && (229)\\ \tau_{i,j} &= \frac{1}{\sqrt{2}}\psi^{-1/2}\gamma_j^{-1/2} && (230)\end{aligned}$$

But τ has a lower bound of $\frac{a_1^2}{a_2^2} f_Z^{-1} + f_D^{-1}$, so it's possible that this has no solution. That would be a state where agents do no learning. Formally,

$$\tau_{i,j} = \max\left(\frac{a_1^2}{a_2^2} f_Z^{-1} + f_D^{-1}, \frac{1}{\sqrt{2}}\psi^{-1/2}\gamma_j^{-1/2}\right) \tag{231}$$

Note that, unlike in the other model, the equilibrium is unique here – all agents individually face a concave problem with an interior solution.

Frequencies with no learning Now using the result for a_1/a_2 from above, at the frequencies where nobody learns, $f_i^{-1} = 0$, we have

$$\begin{aligned} \frac{a_1}{a_2} &= \int_i \gamma_j^{-1}\tau_i^{-1} f_i^{-1} di && (232)\\ &= 0 && (233)\end{aligned}$$

which then implies

$$\tau_{i,j} = \max\left(f_D^{-1}, \frac{1}{\sqrt{2}}\psi^{-1/2}\gamma_j^{-1/2}\right) \tag{234}$$

To get a_2, we have

$$\int_i \left(cj^2 + b\right)\tau_i^{-1}\left(\frac{a_1}{a_2^2}f_Z^{-1} - \tau_i\right)di = -a_2^{-1} \tag{235}$$

$$\gamma_j = a_2 \tag{236}$$

So the sensitivity of the price to supply shocks is increasing in the cost of holding inventory, b, and the trading costs, c. It is also higher at higher frequencies – it is harder to temporarily push through supply than to do it persistently.

Frequencies with learning At the frequencies at which there is learning, where

$$f_D^{-1} < \frac{1}{\sqrt{2}}\psi^{-1/2}\gamma_j^{-1/2} \tag{237}$$

we have, just by rewriting the τ equation,

$$f_i^{-1} = \tau_i - \frac{a_1^2}{a_2^2}f_Z^{-1} - f_D^{-1} \tag{238}$$

Using the second equation from above,

$$\int_i \gamma_j^{-1}\tau_i^{-1}\left(\frac{a_1}{a_2^2}f_Z^{-1} - \tau_i\right)di = -a_2^{-1} \tag{239}$$

$$\int_i \gamma_j^{-1}\tau_i^{-1}\left(\frac{a_1}{a_2}f_Z^{-1} - a_2\tau_i\right)di = -1 \tag{240}$$

$$\int_i \gamma_j^{-1}\left(\tau_i^{-1}\frac{a_1}{a_2}f_Z^{-1} - a_2\right)di = -1 \tag{241}$$

Under the assumption of a symmetric strategy, this is

$$\tau^{-1}\frac{a_1}{a_2}f_Z^{-1} - a_2 = -\gamma_j \tag{242}$$

$$\frac{a_1}{a_2} = \tau f_Z\left(-\gamma_j + a_2\right) \tag{243}$$

Using the other equilibrium condition, we have

$$\begin{aligned}
\int_i \gamma_j^{-1}\tau_i^{-1}\left(\tau_i - \frac{a_1^2}{a_2^2}f_Z^{-1} - f_D^{-1}\right)di &= \frac{a_1}{a_2} && (244)\\
\int_i \gamma_j^{-1}\left(1 - \tau_i^{-1}\frac{a_1}{a_2}f_Z^{-1}\frac{a_1}{a_2} - \tau_i^{-1}f_D^{-1}\right)di &= \frac{a_1}{a_2} && (245)\\
1 - \left(-\gamma_j + a_2\right)\frac{a_1}{a_2} - \tau_i^{-1}f_D^{-1} &= \left(cj^2 + b\right)\frac{a_1}{a_2} && (246)
\end{aligned}$$

$$1 - \tau_i^{-1}f_D^{-1} = a_1 \tag{247}$$

Plugging in the formula for τ_i when there is learning,

$$1 - \sqrt{2}\psi^{1/2}\gamma_j^{1/2}f_D^{-1} = a_1. \tag{248}$$

The expression for a_2 can be obtained from:

$$\frac{a_1}{\tau f_Z} = \left(-\gamma_j + a_2\right)a_2. \tag{249}$$

Since $a_1/\tau f_Z > 0$, we know that there is only one solution to this equation for $a_2 > 0$. The positive root is

$$a_2 = \frac{\gamma_j + \sqrt{\gamma_j^2 + 4\frac{a_1}{\tau f_Z}}}{2} \tag{250}$$

G Results when fundamentals are difference-stationary

In the main text, we assume that the level of fundamentals is stationary. Here we examine an extension in which fundamentals are stationary in terms of first differences and show that the results go through nearly identically, with the primary difference being in how the long-term portfolio is defined.

G.1 Informed investors under difference stationarity

We assume that D_0 is known to investors when making decisions, and without loss of generality normalize $D_0 = 0$. Define Δ to be the first difference operator so that

$$\Delta D_t = D_t - D_{t-1} \tag{251}$$

and define the vector $\Delta D \equiv [\Delta D_1, \Delta D_2, ...\Delta D_T]'$. We assume that

$$\Delta D \sim N\left(0, \Sigma_D\right). \tag{252}$$

For any given allocation to the futures contracts, there is an allocation to claims on ΔD that gives an identical payoff. Specifically, an allocation $Q_i'D$ can be exactly replicated by

$$\begin{aligned} Q_i'D &= Q_i'L_1\Delta D && (253) \\ &= \left(L_1'Q_i\right)'\Delta D && (254) \end{aligned}$$

where L_1 is a matrix that creates partial sums,

$$L_1 \equiv \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \\ 1 & 1 & 1 & \\ \vdots & & & \ddots \end{bmatrix} \tag{255}$$

So an allocation of Q_i to the futures is equivalent to an allocation of $L_1'Q_i$ to claims on the first differences of fundamentals, which we will call the growth rate futures. Define the notation

$$Q_{\Delta,i} \equiv L_1'Q_i \tag{256}$$

Furthermore, the prices of the growth rate futures are simply the vector ΔP (by the law of one price). We can therefore rewrite the optimization problem equivalently as

$$\max T^{-1}\sum_{t=1}^{T}\beta^{t}Q_{\Delta,i,t}E_{0,i}\left[\Delta D_{t}-\Delta P_{t}\right]-\frac{1}{2}\left(\rho T^{-1}\right)Var_{0,i}\left[\sum_{t=1}^{T}\beta^{t}Q_{\Delta,i,t}\left(\Delta D_{t}-\Delta P_{t}\right)\right] \tag{257}$$

Now suppose for the moment that we are able to solve the entire model in terms of first differences (that is not obvious as we will need to ensure that noise trader demand is also difference stationary). So we have an allocation $Q_{\Delta D,i}$. An allocation to the first differences is then equivalent to an allocation of $\left(L_1'\right)^{-1}Q_{\Delta,i}$ to the levels (which follows trivially from the definition of $Q_{\Delta,i}$ in (256)).

Since our maintained assumption is that we will solve the model in first differences in the same way we did in the main text for levels, that means that we will continue to use the rotation Λ, but now in first differences. So the frequency domain allocations in terms of first differences will be

$$\tilde{Q}_{\Delta D,i}=\Lambda\tilde{q}_{\Delta,i} \tag{258}$$

where $\tilde{Q}_{\Delta D,i,t}\equiv Q_{\Delta D,i,t}\beta^{t}$. $\tilde{q}_{\Delta,i}$ now represents the allocations to different frequencies of growth in fundamentals. The key question, then, is what that implies for the behavior of portfolios in terms of levels. We have

$$\begin{aligned}\tilde{Q}_{i} &= \left(L_1'\right)^{-1}\tilde{Q}_{\Delta,i} &(259)\\ &= \left(L_1'\right)^{-1}\Lambda\tilde{q}_{\Delta,i} &(260)\end{aligned}$$

So in terms of levels, the basis vectors, instead of being Λ, are $\left(L_1'\right)^{-1}\Lambda$.

For $\left(L_1'\right)^{-1}$ we have

$$\left(L_1'\right)^{-1}\equiv\begin{bmatrix}1 & -1 & 0 & \cdots & 0\\ 0 & 1 & -1 & & \vdots\\ 0 & 0 & 1 & \ddots & 0\\ \vdots & & & \ddots & -1\\ 0 & \cdots & 0 & 0 & 1\end{bmatrix} \tag{261}$$

So the way that $\left(L_1'\right)^{-1}$ transforms a matrix is to take a forward difference of each column, and then retaining the value of the final row. A way to see the implications of that transformation is to approximate the finite differences of the sines and cosines as derivatives. The columns of $\left(L_1'\right)^{-1}\Lambda$ are equal to $\left(L_1'\right)^{-1}c_j$ and $\left(L_1'\right)^{-1}s_j$, which can be written using standard trigonometric formulas as:

$$\left(L_1'\right)^{-1}c_j \approx \begin{bmatrix} 2\sin\left(\frac{1}{2}\omega_j\right)\sqrt{\frac{2}{T}}\left\{\sin\left(\omega_j\left(t-\frac{1}{2}\right)\right)\right\}_{t=2}^{T} \\ \sqrt{\frac{2}{T}}\cos\left(\omega_j(T-1)\right) \end{bmatrix} \tag{262}$$

$$\left(L_1'\right)^{-1}s_j \approx \begin{bmatrix} -2\sin\left(\frac{1}{2}\omega_j\right)\sqrt{\frac{2}{T}}\left\{\cos\left(\omega_j\left(t-\frac{1}{2}\right)\right)\right\}_{t=2}^{T} \\ \sqrt{\frac{2}{T}}\sin\left(\omega_j(T-1)\right) \end{bmatrix} \tag{263}$$

The column c_j represents a portfolio in terms of the first differences of fundamentals with weights equal to a cosine fluctuating at frequency ω_j. $\left(L_1'\right)^{-1}c_j$ measures the loadings of that portfolio on claims to the level of fundamentals. These loadings also fluctuate at frequency ω_j, with the only difference being the replacement of the cosine with a sine function. (Intuitive, the loadings are approximately equal to the derivative of the columns of Λ with respect to time; taking derivatives does not affect the characteristic frequency of fluctuations.)

So consider a relatively short-term investor, whose portfolio weights are all close to zero except for a large value in the vector $q_{\Delta,i}$ at some large value of j. By assumption, that investor holds a portfolio whose loadings on the first differences of fundamentals fluctuate at frequency ω_j. What the approximations in (262–263) show, though, is that that investor's positions measured in terms of the *level* of fundamentals (i.e. $\tilde{Q}_i$) has loadings that also fluctuate at frequency ω_j.

One subtlety is in the lowest-frequency portfolio, $(L_1'^{-1}\left(\frac{1}{\sqrt{2}}c_0\right)$. That portfolio puts equal weight on growth in fundamentals on all dates – it is a bet on the sample mean growth rate. In terms of levels, note that $\left(L_1'\right)^{-1}\left(\frac{1}{\sqrt{2}}c_0\right) = \left[0,0,0,...,\sqrt{2/T}\right]$. A person who wants to bet on the mean growth rate between dates 1 and T can do that by buying a claim to fundamentals only on date T.[21]

[21] The highest frequency portfolio, $(L_1'^{-1}\left(\frac{1}{\sqrt{2}}c_{\frac{T}{2}}\right)$, is given by $1/\sqrt{T}\,(2,-2,...,2,1)'$, and therefore fluctuates at the highest sample frequency.

G.2 Noise traders under difference stationarity

Last, we need to show that noise trader demand will also take a form such that the entire model can be solved in terms of first differences (and then shifted back into levels for interpretation). First, as above, since the model expressed in first differences is just a linear transformation of the levels, the noise traders' optimization problem can be written in terms of first differences,

$$\max T^{-1} \sum_{t=1}^{T} \beta^t N_{\Delta,t} E_{0,N} \left[\Delta D_t - \Delta P_t\right] - \frac{1}{2}\left(\rho T^{-1}\right) Var_{0,N}\left[\sum_{t=1}^{T} \beta^t N_{\Delta,t}\left(\Delta D_t - \Delta P_t\right)\right] \tag{264}$$

where $N_{\Delta,t}$ is the demand of the noise traders for the claims on first differences.

We assume that the noise traders understand that fundamentals have a unit root and that they therefore have priors and signals that refer to the change in fundamentals. The analogs to (62) and (63) are then

$$\Delta D \sim N\left(0, \Sigma_{N\Delta}^{prior}\right) \tag{265}$$

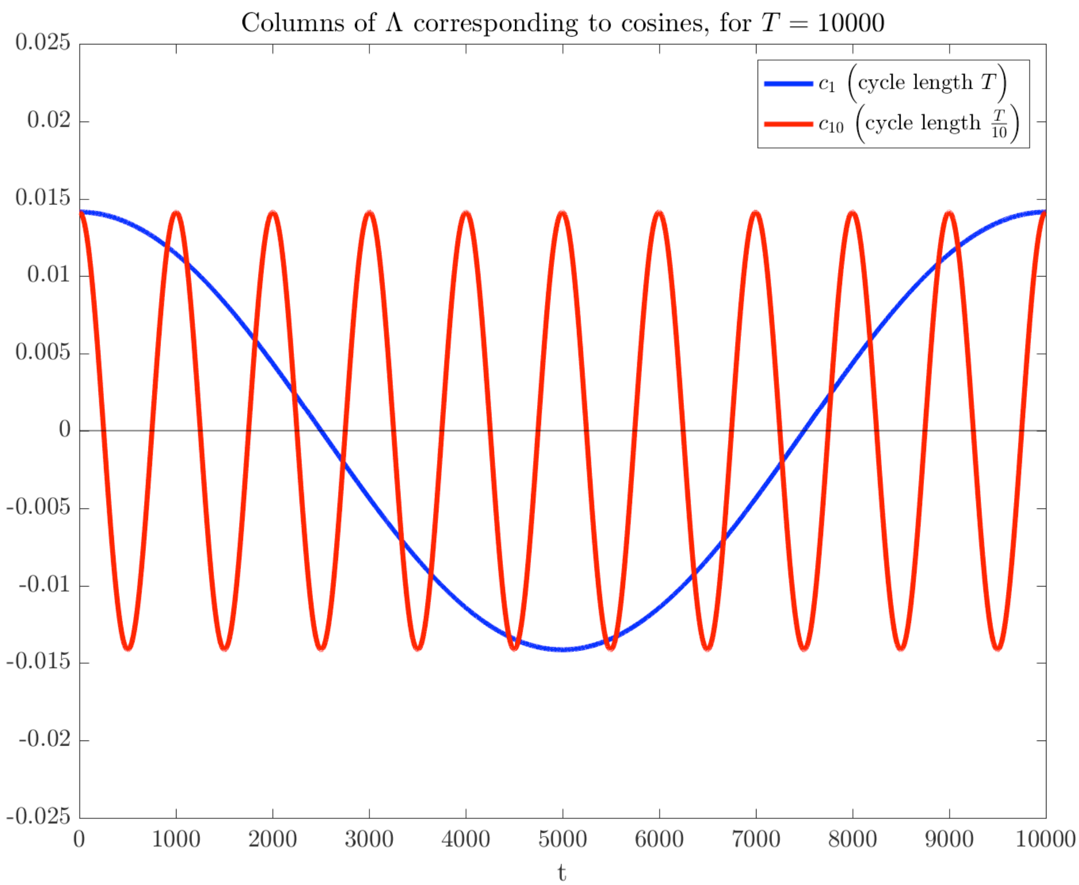
$$S \sim N\left(\Delta D, \Sigma_{N\Delta}^{signal}\right) \tag{266}$$

and the Bayesian update is

$$\Delta D \mid S \sim N\left(\Sigma_{N\Delta}\left(\Sigma_{N\Delta}^{signal}\right)^{-1} S, \Sigma_{N\Delta}\right) \tag{267}$$

$$\text{where } \Sigma_{N\Delta} \equiv \left(\left(\Sigma_{N\Delta}^{signal}\right)^{-1} + \left(\Sigma_{N\Delta}^{prior}\right)^{-1}\right)^{-1} \tag{268}$$

Figure 1: Portfolio weights



Notes: Portfolio weights for the cosing frequency portfolios c1 and c10, as defined in the main text. The horizontal axis is time, or the maturity of the corresponding futures contract. The vertical axis is the weight which each portfolio puts on that futures contract.